The Cubic Equation

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As is well known, equations of degree up to 4 can be ≪solved in radicals≫. The solutions can be obtained, apart from the usual arithmetic operations, by the extraction of roots. In the case of the quadratic equation, this has a very concrete meaning. Even if the coefficients are arbitrary *complex* numbers, the solutions can always be calculated by the extraction of roots from nonnegative *real* numbers. This can, if necessary, even be done by hand.

It is therefore important to emphasize that, already in the case of the cubic equation with only real coefficients, ≪solvability in radicals≫ means much less. Whenever there are three distinct real solutions, calculating them involves finding a solution to the equation

$$z^3 = u$$

for some complex, nonreal u. (To be quite precise, this is not necessary if coefficients as well as solutions are rational.) Most of the time this cannot be done by just extracting roots from nonnegative real numbers. Therefore, while the question of whether or not a certain equation can be solved in radicals has had a profound influence on the development of mathematics, it is not of any decisive importance to the actual search for solutions.

1 Arbitrary complex coefficients

As any cubic equation $z^3 + az^2 + bz + c = 0$ can be transformed to the form

$$z^3 + pz + q = 0 \tag{1}$$

by the substitution of z with $z - \frac{a}{3}$, solving the cubic boils down to solving equation (1). The identity

$$(u+v)^3 + p(u+v) + q = (u^3 + v^3 + q) + (3uv + p)(u+v) \tag{2}$$

connects (1) with the system

$$\left\{ \begin{array}{rcl} 3uv &=& -p, \\ u^3 + v^3 &=& -q. \end{array} \right. \tag{3}$$

The idea of letting $z = u + v$ apparently goes back to the Italien mathematician Tartaglia (1499 or 1500–1557). However, while to Tartaglia all numbers were real (and even nonnegative), we shall admit arbitrary complex numbers.

Theorem 1

(i) If (u_1, v_1) is a solution of (3), then $u_1 + v_1$ is a solution of (1).

(ii) If $u_1 + v_1$ is a solution of (1) and $3u_1v_1 + p = 0$, then (u_1, v_1) is a solution of (3).

(iii) If z_1 is a solution of (1), then there is a solution (u_1, v_1) of (3) such that $z_1 = u_1 + v_1$. Apart from order, u_1 and v_1 are uniquely determined:

$$u_1 = \frac{z_1}{2} + \varepsilon, \quad v_1 = \frac{z_1}{2} - \varepsilon$$

for some ε such that $\varepsilon^2 = \left(\frac{z_1}{2}\right)^2 + \frac{p}{3}$.

Proof: (i) and (ii) are immediate consequences of (2).
(iii) The conditions $u_1 + v_1 = z_1$ and $u_1 v_1 = -\frac{p}{3}$ uniquely determine (apart from order) u_1, v_1 as the solutions of the equation $w^2 - z_1 \cdot w - \frac{p}{3} = 0$. Therefore $u_1 = \frac{z_1}{2} + \varepsilon$, $v_1 = \frac{z_1}{2} - \varepsilon$ for some ε such that $\varepsilon^2 = \left(\frac{z_1}{2}\right)^2 + \frac{p}{3}$. By (ii), (u_1, v_1) is a solution of (3). q.e.d.

The first equation of (3) implying $u^3 v^3 = \left(-\frac{p}{3}\right)^3$, (3) implies that u^3, v^3 are the solutions of the equation $w^2 + q \cdot w - \left(\frac{p}{3}\right)^3 = 0$. These solutions are $-\frac{q}{2} + \delta$ and $-\frac{q}{2} - \delta$ for some δ such that $\delta^2 = \Delta$, where

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

Therefore, (3) implies

$$\begin{cases} 3uv = -p, \\ u^3 = -\frac{q}{2} + \delta \text{ and } v^3 = -\frac{q}{2} - \delta \text{ for some } \delta \text{ such that } \delta^2 = \Delta. \end{cases} \tag{4}$$

The converse is trivially true, hence we get:

Lemma 1.1 *The equation systems (3) and (4) are equivalent.*

Next we prove:

Theorem 2 *Let δ, u_1 be such that $\delta^2 = \Delta$, $u_1^3 = -\frac{q}{2} + \delta$, where $u_1 \neq 0$, and let $v_1 = -\frac{p}{3u_1}$. Then*

(i) $v_1^3 = -\frac{q}{2} + \delta$,

(ii) $u_1 + v_1$ *is a solution of (1).*

Proof: $\delta^2 = \Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$ implies that $\left(-\frac{p}{3}\right)^3 = \left(\frac{q}{2}\right)^2 - \delta^2 = \left(-\frac{q}{2} + \delta\right)\left(-\frac{q}{2} - \delta\right) = u^3\left(-\frac{q}{2} - \delta\right)$. From the definition of v_1 we get $v_1^3 = \left(-\frac{p}{3}\right)^3 \cdot \frac{1}{u_1^3} = u_1^3\left(-\frac{q}{2} - \delta\right) \cdot \frac{1}{u_1^3} = -\frac{q}{2} - \delta$. So (u_1, v_1) is a solution of (4) and, by Lemma 1.1, of (3). By (i) of Theorem 1, $u_1 + v_1$ is a solution of (1). q.e.d.

This theorem is, in a way, an instruction on how to obtain a solution z_1:

- Find a δ such that $\delta^2 = \Delta$ (can be done by extraction of roots from nonnegative real numbers).
- Find a solution u_1 to the equation $u^3 = -\frac{q}{2} + \delta$ (by de Moivre's formula).
- Calculate $v_1 = -\frac{p}{3u_1}$.
- Let $z_1 = u_1 + v_1$.

We let ζ be one of the nonreal solutions of the equation $x^3 = 1$, let's say

$$\zeta = \frac{1}{2}\left(-1 + \sqrt{3} \cdot \mathrm{i}\right). \tag{5}$$

As $z^3 - 1 = (z-1)(z^2 + z + 1)$, ζ satisfies

$$\zeta^2 + \zeta + 1 = 0. \tag{6}$$

If (u_1, v_1) is a solution of (3), then obviously, $(\zeta^k u_1, \zeta^{-k} v_1)$ and $(\zeta^{-k} v_1, \zeta^k u_1)$ also are ($k \in \mathbb{Z}$). We now prove the converse:

Theorem 3 *If (u_1, v_1) and (u_2, v_2) are both solutions of (3), then*

$$(u_2, v_2) = (\zeta^k u_1, \zeta^{-k} v_1) \qquad \text{or} \qquad (u_2, v_2) = (\zeta^{-k} v_1, \zeta^k u_1)$$

for some $k \in \{0, 1, 2\}$.

Proof: We first assume that $p = 0$. Then, e.g., $u_1 = v_2 = 0$. From the second equation of (3) follows $v_1^3 = -q = u_2^3$. So $u_2 = \zeta^k v_1$ for some $k \in \{0, 1, 2\}$. Trivially, $v_2 = \zeta^{-k} u_1$. We let $k' = 3 - k \bmod 3$. Then $\zeta^{k'} = \zeta^{-k}$ and $\zeta^{-k'} = \zeta^k$, therefore $(u_2, v_2) = (\zeta^{-k'} v_1, \zeta^{k'} u_1)$, where $k' \in \{0, 1, 2\}$. All other cases are treated similarly.

Now we assume $p \neq 0$. Then Lemma 1.1 implies that for some δ such that $\delta^2 = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$, $u_1^3 = -\frac{q}{2} + \delta$, and either $u_2^3 = -\frac{q}{2} + \delta$ or $v_2^3 = -\frac{q}{2} + \delta$. In the first case, $u_2^3 = u_1^3$, and therefore $u_2 = \zeta^k u_1$ for some $k \in \{0, 1, 2\}$. The first equation of (3) implies that $u_2 v_2 = u_1 v_1 = -p$, therefore $\zeta^k u_1 v_2 = u_1 v_1$. As $p \neq 0$, $u_1 \neq 0$, and we conclude that $\zeta^k v_2 = v_1$, and therefore $v_2 = \zeta^{-k} v_1$. In the second case, $v_2^3 = u_1^3$, and therefore $v_2 = \zeta^k u_1$ for some $k \in \{0, 1, 2\}$. As before, $u_2 v_2 = u_2 \zeta^k u_1 = u_1 v_1 = -p$, and $u_2 \zeta^k = v_1$, $u_2 = \zeta^{-k} v_1$. q.e.d.

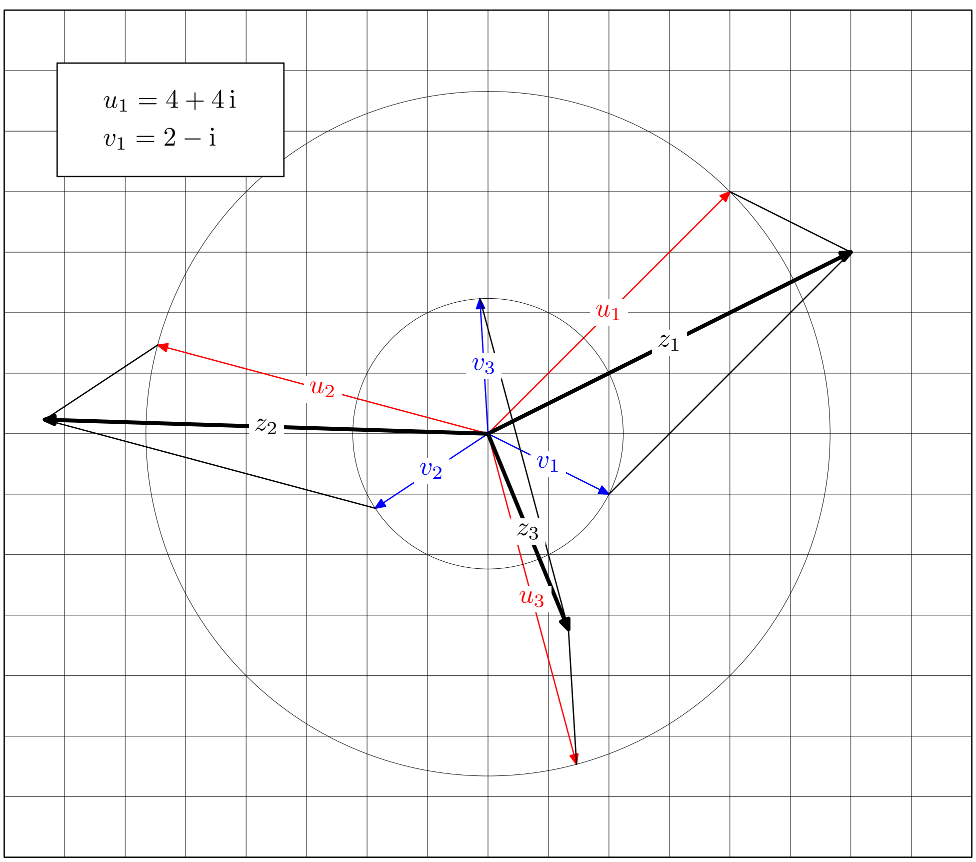
Theorem 4 *Let z_1, z_2, z_3 be the (not necessarily distinct) solutions of (1), and let (u_1, v_1) be any solution of (3). Then for $k = 0, 1, 2$, the values $\zeta^k u_1 + \zeta^{-k} v_1$ are a permutation of the solutions z_1, z_2, z_3.*

Proof: By *(iii)* of Theorem 1, each of the solutions z_1, z_2, z_3 is of the form $u' + v'$, where (u', v') is a solution of (3). By Theorem 3, $(u', v') = (\zeta^k u_1, \zeta^{-k} v_1)$ or $(u', v') = (\zeta^{-k} v_1, \zeta^k u_1)$, and therefore $u' + v' = \zeta^k u_1 + \zeta^{-k} v_1$ for some $k \in \{0, 1, 2\}$. q.e.d.

Example 1 The following picture illustrates how the solution of the equation $z^3 - (36 + 12\,\mathrm{i})z + (126 - 117\,\mathrm{i}) = 0$ is reduced to the solution of the simpler equations

$$u^3 = -\frac{q}{2} + \delta = -128 + 128\,\mathrm{i}, \quad v^3 = -\frac{q}{2} - \delta = 2 - 11\,\mathrm{i},$$

where $\delta = -65 + \frac{139}{2}\,\mathrm{i}$, $\delta^2 = \Delta = -\frac{2421}{4} - 9035\,\mathrm{i}$. We can take as a start $u_1 = 4 + 4\,\mathrm{i}$ and $v_1 = 2 - \mathrm{i}$.



We then get u_2, u_3 by twice rotating u_1 by $\frac{2\pi}{3}$ (twice multiplying u_1 by ζ), and v_2, v_3 by twice rotating v_1 by $-\frac{2\pi}{3}$ (twice multiplying v_1 by ζ^{-1}).

Theorem 5 *If z_1, z_2 and z_3 are the solutions of (1), then*

$$\Delta = -\frac{1}{108}\left((z_2 - z_1)(z_3 - z_2)(z_1 - z_3)\right)^2.$$

Therefore

(i) $\Delta \neq 0$ if and only if all solutions are distinct.

(ii) If z_1, z_2, z_3 are all distinct and real, then Δ is also real, and $\Delta < 0$.

PROOF: In view of Theorem 4 we may assume that $z_1 = u_1 + v_1$, $z_2 = \zeta u_1 + \zeta^{-1} v_1$, $z_3 = \zeta^2 u_1 + \zeta^{-2} v_1$ for some solution (u_1, v_1) of (3). We find

$$\begin{aligned} z_2 - z_1 &= (\zeta - 1)u_1 + (\zeta^2 - 1)v_1 &&= (\zeta - 1)(u_1 + (\zeta + 1)v_1) &&= (\zeta - 1)(u_1 - \zeta^2 v_1), \\ z_1 - z_3 &= (1 - \zeta^2)u_1 + (1 - \zeta)v_1 &&= (1 - \zeta)((1 + \zeta)u_1 + v_1) &&= (\zeta - 1)(\zeta^2 u_1 - v_1) \end{aligned}$$

and therefore, applying $(\zeta^2 - \zeta)^2 = \zeta^4 - 2\zeta^3 + \zeta^2 = (\zeta + \zeta^2) - 2 = -1 - 2 = -3$,

$$\begin{aligned} (z_2 - z_1)(z_1 - z_3) &= (\zeta - 1)^2(u_1 - \zeta^2 v_1)(\zeta^2 u_1 - v_1) &&= (\zeta - 1)^2(\zeta^2 u_1^2 - \zeta u_1 v_1 - u_1 v_1 + \zeta^2 v_1^2) \\ &= (\zeta - 1)^2(\zeta^2 u_1^2 + \zeta^2 u_1 v_1 + \zeta^2 v_1^2) &&= (\zeta^2 - \zeta)^2(u_1^2 + u_1 v_1 + v_1^2) \\ &= -3(u_1^2 + u_1 v_1 + v_1^2). \end{aligned}$$

We further have

$$z_3 - z_2 = (\zeta^2 - \zeta)u_1 + (\zeta - \zeta^2)v_1 = (\zeta^2 - \zeta)(u_1 - v_1)$$

and therefore $(z_2 - z_1)(z_3 - z_2)(z_1 - z_3) = -3(u_1^2 + u_1 v_1 + v_1^2) \cdot (\zeta^2 - \zeta)(u_1 - v_1) = -3(\zeta^2 - \zeta)(u_1^3 - v_1^3)$. By Lemma 1.1, (u_1, v_1) is also a solution of (4), therefore $u_1^3 = -\frac{q}{2} + \delta$, $v_1^3 = -\frac{q}{2} - \delta$ and $u_1^3 - v_1^3 = 2\delta$ for some δ such that $\delta^2 = \Delta$. Thus finally, $\left((z_2 - z_1)(z_3 - z_2)(z_1 - z_3)\right)^2 = 9(-3) \cdot 4\Delta = -108\,\Delta$. q.e.d.

2 All coefficients real

While the above results hold for arbitrary complex coefficients, we now retrict ourselves to real values of p and q. Then $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$ also is real, and there are three possibilities:

I. $\Delta = 0$ («vanishing discriminant»),

II. $\Delta > 0$ («classical case»),

III. $\Delta < 0$ («irreducibel case»).

We will show that in cases I and III, there are only real solutions, whereas in case II, there are two nonreal solutions (which then are conjugate complex).

2.1 The case of the vanishing discriminant

Let ρ be the uniquely determined real number such that

$$\rho^3 = \frac{q}{2}.$$

Then $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = 0$ implies $\rho^6 + \left(\frac{p}{3}\right)^3 = 0$, $\left(\frac{p}{3}\right)^3 = -\rho^6$, $\frac{p}{3} = -\rho^2$. Thus we have $p = -3\rho^2$, $q = 2\rho^3$, and (1) becomes

$$x^3 - 3\rho^2 \cdot x + 2\rho^3 = 0.$$

Obviously, one of the solutions is $z_1 = \rho$, and after splitting off the factor $x - \rho$, for z_2, z_3 we get the equation $x^2 + \rho \cdot x - 2\rho^2 = 0$ having the solutions ρ and -2ρ.

Theorem 6 *If p, q are real and $\Delta = 0$, then (1) has the solutions $\sqrt[3]{\frac{q}{2}}$, $\sqrt[3]{\frac{q}{2}}$. $-2\sqrt[3]{\frac{q}{2}}$.*

2.2 The classical case

If p and q are real, and $\Delta > 0$, equation (1) has one real and two nonreal solutions, the latter being conjugate complex. We start with the following

Lemma 2.1 *If z_1 is a solution, then (1) is equivalent to $(x - z_1)(x^2 + z_1 \cdot x + (z_1^2 + p)) = 0$, and*

$$\Delta = \frac{1}{108}\left(3z_1^2 + 4p\right)\left(3z_1^2 + p\right)^2 = -\frac{1}{108}\Delta_1\left(3z_1^2 + p\right)^2,$$

where $\Delta_1 = -\left(3z_1^2 + 4p\right)$ is the discriminant of the quadratic factor.

Proof: From $z_1^3 + pz_1 + q = 0$, we get $\frac{q}{2} = -\frac{1}{2}\left(z_1^3 + pz_1\right)$ and therefore $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \frac{1}{4}\left(z_1^3 + pz_1\right)^2 + \frac{1}{27}p^3 = \frac{1}{108}\left(27z_1^6 + 54pz_1^4 + 27p^2z_1^2 + 4p^3\right) = \frac{1}{108}\left(3z_1^2 + 4p\right)\left(9z_1^4 + 6pz_1^2 + p^2\right)$, thus $\Delta = \frac{1}{108}\left(3z_1^2 + 4p\right)\left(3z_1^2 + p\right)^2$. For the discriminant of the quadratic factor, we find $\Delta_1 = z_1^2 - 4(z_1^2 + p) = -3z_1^2 - 4p = -(3z_1^2 + 4p)$. Hence $\Delta = -\frac{1}{108}\Delta_1\left(3z_1^2 + p\right)^2$. q.e.d.

We define two real numbers u_1 and v_1, letting

$$u_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}}, \quad v_1 = \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}. \tag{7}$$

Theorem 7

(i) (u_1, v_1) is a solution of (3).

(ii) $z_1 = u_1 + v_1$ is a solution of (1).

(iii) z_1 is the only real solution of (1). The other solutions are z_2, $z_3 = \frac{1}{2}\left(-(u_1 + v_1) \pm \sqrt{3}\,(u_1 - v_1)\cdot \mathrm{i}\right)$.

Proof: *(i)* $3u_1v_1 = 3\sqrt[3]{\left(-\frac{q}{2} + \sqrt{\Delta}\right)\left(-\frac{q}{2} + \sqrt{\Delta}\right)} = 3\sqrt[3]{\left(\frac{q}{2}\right)^2 - \Delta} = 3\sqrt[3]{\left(\frac{q}{2}\right)^2 - \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3} = 3\sqrt[3]{\left(-\frac{p}{3}\right)^3}$, hence $3u_1v_1 = -p$. $u_1^3 + v_1^3 = \left(-\frac{q}{2} + \sqrt{\Delta}\right) + \left(-\frac{q}{2} - \sqrt{\Delta}\right) = -q$.

(ii) As (u_1, v_1) is a solution of (3), $u_1 + v_1$ is a solution of (1) by *(i)* of Theorem 1.

(iii) By Lemma 2.1, z_2, z_3 are the solutions of the quadratic equation $x^2 + z_1 \cdot x + (z_1^2 + p) = 0$ having the discriminant $\Delta_1 = -(3z_1^2 + 4p)$. On the other hand, from *(ii)* we get $z_1^2 - (u_1 - v_1)^2 = (u_1 + v_1)^2 - (u_1 - v_1)^2 = 4u_1v_1 = -\frac{4p}{3}$, hence $z_1^2 + \frac{4p}{3} = (u_1 - v_1)^2$ and $\Delta_1 = -3(z_1^2 + \frac{4p}{3}) = -3(u_1 - v_1)^2$. Therefore z_2, $z_3 = \frac{1}{2}\left(-z_1 \pm \sqrt{3}\,(u_1 - v_1)\cdot \mathrm{i}\right)$, where $z_1 = u_1 + v_1$. q.e.d.

Of course, Theorem 7 yields the same result as Theorem 6 if $\Delta = 0$.

Example 2 Applying Theorem 7 to the equation $x^3 + x - 2 = 0$, we obtain

$$z_1 = \sqrt[3]{1 + \frac{2}{9}\sqrt{21}} + \sqrt[3]{1 - \frac{2}{9}\sqrt{21}}.$$

Now this seems rather surprising, because obviously, one of the solutions is $z_1 = 1$. This is the only real solution, as $x^3 + x - 2 = (x - 1)(x^2 + x + 2)$, the second factor having no real zeroes. So we are forced to conclude that

$$\sqrt[3]{1 + \frac{2}{9}\sqrt{21}} + \sqrt[3]{1 - \frac{2}{9}\sqrt{21}} = 1.$$

If there are rational numbers a and b such that $\left(a + b\sqrt{21}\right)^3 = 1 + \frac{2}{9}\sqrt{21}$, then $\left(a - b\sqrt{21}\right)^3 = 1 - \frac{2}{9}\sqrt{21}$, and the left side reduces to $(a + b\sqrt{21}) + (a - b\sqrt{21}) = 2a$. We therefore look for b such that $\left(\frac{1}{2} + b\sqrt{21}\right)^3 = 1 + \frac{2}{9}\sqrt{21}$. The conditions for b are

$$\begin{cases} \frac{1}{8} + \frac{63}{2}b^2 &= 1 \\ \frac{3b}{4}\left(1 + 28b^2\right) &= \frac{2}{9}. \end{cases}$$

This system has a unique solution: $b = \frac{1}{6}$. So the sum of the two third roots reduces to

$$\left(\frac{1}{2} + \frac{1}{6}\sqrt{21}\right) + \left(\frac{1}{2} - \frac{1}{6}\sqrt{21}\right) = 1.$$

We might have reached this insight easier with the shortcut furnished by (*iii*) of Theorem 1. Knowing that $z_1 = 1$, we get $\varepsilon = \sqrt{\left(\frac{1}{2}\right)^2 + \frac{1}{3}} = \frac{1}{6}\sqrt{21}$, $u_1 = \frac{1}{2} + \frac{1}{6}\sqrt{21}$, $v_1 = \frac{1}{2} - \frac{1}{6}\sqrt{21}$. For $u_1 - v_1$ we get $\frac{1}{3}\sqrt{21}$ and finally, applying (*iii*) of Theorem 7, $z_1, z_2 = \frac{1}{2}\left(-1 \pm \sqrt{7}\,\mathrm{i}\right)$.

Example 3 $x^3 - x - 1 = 0$. By Theorem 7, we get $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \frac{23}{108} = \frac{1}{18}\sqrt{69}$ and

$$z_1 = \sqrt[3]{\frac{1}{2} + \frac{1}{18}\sqrt{69}} + \sqrt[3]{\frac{1}{2} - \frac{1}{18}\sqrt{69}} \approx 1.3247.$$

Any rational solution would have to be an integer, and any integer solution would have to be a divisor of $q = -1$, but 1 and -1 are not solutions. So there are no rational solutions. Consequently, there is no use looking for rational r, s such that $\left(r + s\sqrt{69}\right)^3 = \frac{1}{2} + \frac{1}{18}\sqrt{69}$. If there were, this would imply $\left(r - s\sqrt{69}\right)^3 = \frac{1}{2} - \frac{1}{18}\sqrt{69}$, and we would have the rational solution $z_1 = (r + s\sqrt{69}) + (r - s\sqrt{69}) = 2r$.

Theorem 8 *Let z_1, z_2, z_3 be the solutions of the equation $x^3 + px + q = 0$ where z_1 and b are real numbers such that $b > 0$ and $x_{2,3} = -\frac{z_1}{2} \pm b \cdot \mathrm{i}$. Then if $z_1 = u_1 + v_1$,*

$$u_1, v_1 = \frac{z_1}{2} \pm \frac{b}{\sqrt{3}}.$$

PROOF: The equation corresponding to the given solutions is $x^3 + \left(b^2 - \frac{3}{4}z_1^2\right)x + q = 0$. From part (*iii*) of Theorem 1 follows $u_1, v_1 = \frac{z_1}{2} \pm \varepsilon$, where $\varepsilon = \sqrt{\left(\frac{z_1}{2}\right)^2 + \frac{p}{3}} = \sqrt{\frac{z_1^2}{4} + \frac{b^2}{3} - \frac{z_1^2}{4}} = \frac{b}{\sqrt{3}}$. q.e.d.

Example 4 $x^3 + 6x - 20 = 0$. We get $\Delta = 108$. As $z_1 = 2$ is a solution, applying Theorem 7 (*ii*), we conclude that

$$\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = 2.$$

Part (*iii*) of Theorem 1 yields $u_1, v_1 = 1 \pm \varepsilon$, where $\varepsilon = \sqrt{\left(\frac{z_1}{2}\right)^2 + \frac{p}{3}} = \sqrt{1+2} = \sqrt{3}$. Hence $u_1, v_1 = 1 \pm \sqrt{3}$. We indeed find

$$(1 \pm \sqrt{3})^3 = 10 \pm 6\sqrt{3}.$$

Splitting off the factor $x - 2$, we find $x_{2,3} = -1 \pm 3\,i$. Using Theorem 8, we again find $u_1, v_1 = 1 \pm \frac{3}{\sqrt{3}} = 1 \pm \sqrt{3}$.

2.3 The irreducibel case

If p, q are real, then $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0$ implies $p < 0$, $p = -|p|$, and $\Delta = -\left(\left(\frac{|p|}{3}\right)^3 - \left(\frac{q}{2}\right)^2\right)$. We let $\delta = \mathrm{i} \cdot \sqrt{-\Delta}$. Then $\left|\left|-\frac{q}{2} + \delta\right|\right| = \left|\left|-\frac{q}{2} + \mathrm{i} \cdot \sqrt{-\Delta}\right|\right| = \sqrt{\left(\frac{q}{2}\right)^2 - \Delta}$, hence

$$\left|\left|-\frac{q}{2} + \delta\right|\right| = \sqrt{\left(\frac{|p|}{3}\right)^3} \neq 0. \tag{8}$$

Therefore the equation $u^3 = -\frac{q}{2} + \delta$ has three distinct solutions u_1, u_2, u_3 such that

$$||u_k|| = \sqrt{\frac{|p|}{3}} \neq 0.$$

We let $v_k = -\frac{p}{3u_k}$ and $x_k = u_k + v_k$ $(i = 1, 2, 3)$. Then by Theorem 2, z_1, z_2, z_3 are solutions of (1). From $u_k v_k = -\frac{p}{3} = \frac{|p|}{3}$ and $u_k\overline{u_k} = ||u_k||^2 = \left(\sqrt{\frac{|p|}{3}}\right)^2 = \frac{|p|}{3} = u_k v_k$, we conclude that $v_k = \overline{u_k}$ and

$$x_k = u_k + \overline{u_k} = 2\,\mathrm{Re}(u_k) = 2\,||u_k||\cos(\arg(u_k)).$$

But $\arg(u_k) = \frac{1}{3}\arg(-\frac{q}{2}+\delta) + (k-1)\cdot\frac{2\pi}{3}$, and $\arg(-\frac{q}{2}+\delta) = \arg(-\frac{q}{2} + \mathrm{i}\cdot\sqrt{-\Delta})$, where $\sqrt{-\Delta} > 0$. Therefore $\varphi = \arg(-\frac{q}{2}+\delta) = \arccos\left(\frac{-\frac{q}{2}}{\left|\left|-\frac{q}{2}+\delta\right|\right|}\right) = \arccos\left(-\frac{q}{2}\sqrt{\left(\frac{3}{|p|}\right)^3}\right)$ by (8). By (*i*) of Theorem 5, all solutions are distinct. We can also see this directly, as $0 < \varphi < \pi$, and therefore $0 < \frac{\varphi}{3} < \frac{\pi}{3}$, $\frac{2\pi}{3} < \frac{\varphi}{3} + \frac{2\pi}{3} < \pi$, $\frac{4\pi}{3} < \frac{\varphi}{3} + \frac{4\pi}{3} < \frac{5\pi}{3}$. As a consequence,

$$-1 < \cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) < -\frac{1}{2} < \cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) < \frac{1}{2} < \cos\frac{\varphi}{3} < 1. \tag{9}$$

This implies that $\mathrm{Re}(u_1)$, $\mathrm{Re}(u_2)$, $\mathrm{Re}(u_3)$, and therefore z_1, z_2, z_3 are all distinct.

Theorem 9 *If $\Delta < 0$, the equation (1) has three distinct real solutions:*

$$x_k = 2\sqrt{\frac{|p|}{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\cdot\sqrt{\left(\frac{3}{|p|}\right)^3}\right) + (k-1)\cdot\frac{2\pi}{3}\right) \qquad (k = 1,\, 2,\, 3).$$

We might add that our deduction of course garanties that in this result, arccos is always defined. More directly, as $p < 0$, the following are equivalend: $\Delta \gtreqless 0$, $\left(\frac{q}{2}\right)^2 - \left(\frac{|p|}{3}\right)^3 \gtreqless 0$, $\left(\frac{q}{2}\right)^2 \gtreqless \left(\frac{|p|}{3}\right)^3$, hence we have:

Lemma 2.2 *If $p < 0$, then $\Delta \gtreqless 0$ if and only if* $\left|-\frac{q}{2}\cdot\sqrt{\left(\frac{3}{|p|}\right)^3}\right| \gtreqless 1.$

2.4 Another, most elegant formulation

In view of the result of Theorem 9, we let

$$z = 2\sqrt{\frac{|p|}{3}}w. \tag{10}$$

Then (1) transforms into $8\sqrt{\left(\frac{|p|}{3}\right)^3}w^3 + 2p\sqrt{\frac{|p|}{3}}w + q = 0$. As $p = 3\,\mathrm{sgn}(p)\frac{|p|}{3}$, this is equivalent to $8\sqrt{\left(\frac{|p|}{3}\right)^3}w^3 + 6\,\mathrm{sgn}(p)\sqrt{\left(\frac{|p|}{3}\right)^3}w = -q$ and, if $p \neq 0$, to

$$4w^3 + 3\,\mathrm{sgn}(p)w = C, \text{ where } C = \frac{-\frac{q}{2}}{\sqrt{\left(\frac{|p|}{3}\right)^3}} = -\frac{q}{2}\sqrt{\left(\frac{3}{|p|}\right)^3}. \tag{11}$$

Therefore, (1) is equivalent to

I. $4w^3 + 3w = C$, if $p > 0$,

II. $4w^3 - 3w = C$, if $p < 0$.

These equations match in form with the identities

$$\begin{cases} 4\sinh^3\varphi & + & 3\sinh\varphi & = & \sinh 3\varphi, \\ 4\cosh^3\varphi & - & 3\cosh\varphi & = & \cosh 3\varphi, \\ 4\cos^3\varphi & - & 3\cos\varphi & = & \cos 3\varphi, \end{cases} \tag{12}$$

whereby we have to keep in mind that the ranges of sinh, cosh, cos are $\mathbb{R}$, $\{x \in \mathbb{R}/x \geq 1\}$, and $\{x \in \mathbb{R}/|x| \leq 1\}$, respectively. Comparing (11) with (12, we get

Theorem 10

(i) $w_1 = \sinh\left(\frac{1}{3}\operatorname{arsinh} C\right)$ *if* $p > 0$,

(ii) $w_1 = \begin{cases} \cosh\left(\frac{1}{3}\operatorname{arcosh} C\right) & \text{if } p < 0 \text{ and } C \geq 1 \\ -\cosh\left(\frac{1}{3}\operatorname{arcosh}(-C)\right) & \text{if } p < 0 \text{ and } C \leq -1 \end{cases}$,

(iii) $w_k = \cos\left(\frac{1}{3}\arccos C + (k-1)\cdot\frac{2\pi}{3}\right)$, *where* $k = 1, 2, 3$ *if* $p < 0$ *and* $|C| \leq 1$.

By Lemma 2.2, (*i*) and (*ii*) correspond to the ≪classical case≫, while (*iii*) is the result in the ≪irreducibel case≫ (Theorem 9). There is, however, a difference. While the deduction of Theorem 9 is based on complex numbers, the deduction of part (*iii*) of Theorem 10, based on the formula $\cos 3\varphi = 4\cos\varphi^3 - 3\cos\varphi$, lies wholly within the realm of the real numbers.

In order to confirm that cases (*i*) and (*ii*) of Theorem 10 lead to the result of Theorem 7, we first prove

Lemma 2.3 $$\sqrt{\left(\frac{3}{|p|}\right)^3}\left(-\frac{q}{2}+\sqrt{\Delta}\right) = \begin{cases} C + \sqrt{C^2+1} & \text{if } p > 0, \\ C + \sqrt{C^2-1} & \text{if } p < 0 \text{ and } C^2 \geq 1. \end{cases}$$

Proof: If $p > 0$, then $C^2 + 1 = \frac{\left(\frac{q}{2}\right)^2}{\left(\frac{p}{3}\right)^3} + 1 = \frac{\Delta}{\left(\frac{p}{3}\right)^3}$, therefore $C + \sqrt{C^2+1} = \frac{1}{\sqrt{\left(\frac{p}{3}\right)^3}}\left(-\frac{q}{2}+\sqrt{\Delta}\right) = \sqrt{\left(\frac{3}{|p|}\right)^3}\left(-\frac{q}{2}+\sqrt{\Delta}\right)$. If $p < 0$ and $C^2 \geq 1$, then $C^2 - 1 = \frac{\left(\frac{q}{2}\right)^2}{\left(-\frac{p}{3}\right)^3} - 1 = \frac{\Delta}{\left(-\frac{p}{3}\right)^3} \geq 0$, therefore $C + \sqrt{C^2-1} = \frac{1}{\sqrt{\left(-\frac{p}{3}\right)^3}}\left(-\frac{q}{2}+\sqrt{\Delta}\right) = \sqrt{\left(\frac{3}{|p|}\right)^3}\left(-\frac{q}{2}+\sqrt{\Delta}\right)$. q.e.d.

Theorem 11 *If* $p > 0$, *or* $p < 0$ *and* $C^2 \geq 1$, *then (i) and (ii) of Theorem 10 both lead to*

$$z_1 = 2\sqrt{\frac{|p|}{3}}\,w_1 = \sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}}.$$

Proof: (*i*). $\frac{1}{3}\operatorname{arsinh} C = \ln\sqrt[3]{C+\sqrt{C^2+1}}$, hence $w_1 = \sinh\left(\frac{1}{3}\operatorname{arsinh} C\right) = \frac{1}{2}\left(\sqrt[3]{C+\sqrt{C^2+1}} - \frac{1}{\sqrt[3]{C+\sqrt{C^2+1}}}\right)$.
By Lemma 2.3, $\sqrt[3]{C+\sqrt{C^2+1}} = \sqrt{\frac{3}{p}}\sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}}$, $\frac{1}{\sqrt[3]{C+\sqrt{C^2+1}}} = \sqrt{\frac{p}{3}}\sqrt[3]{\frac{-\frac{q}{2}-\sqrt{\Delta}}{\left(-\frac{p}{3}\right)^3}} = \sqrt{\frac{p}{3}}\left(-\frac{p}{3}\right)\sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}} = -\sqrt{\frac{3}{p}}\sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}}$, therefore $w_1 = \frac{1}{2}\sqrt{\frac{3}{p}}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}}\right)$, and $z_1 = 2\sqrt{\frac{p}{3}}\,w_1 = \sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}}$.

(*ii*). The proof runs analogously to that of case (*i*), using that for $C \geq 1$, $\frac{1}{3}\arccos C = \ln\sqrt[3]{C+\sqrt{C^2-1}}$ and thus

$$\cosh\left(\frac{1}{3}\arccos C\right) = \frac{1}{2}\left(\sqrt[3]{C+\sqrt{C^2-1}} + \frac{1}{\sqrt[3]{C+\sqrt{C^2-1}}}\right),$$

where, by Lemma 2.3, $\sqrt[3]{C+\sqrt{C^2-1}} = \sqrt{\frac{3}{|p|}}\sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}}$.

If $C \leq -1$, the equation $4w^3 - 3w = C$ has to be replaced by the equivalent equation $4(-w)^3 - 3(-w) = -C \geq 1$. q.e.d.

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