Sudoku - A Tutorial

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June 17, 2014 - Version 2

Contents

This second version corrects the definition of the x_1-chain. In the first version (of november 5th 2011), x_1-chains were restricted to chains of an odd number of strong edges. But obviously, the decisive property is preserved if any two x_1-chains are connected by a weak edge.

I am very grateful to Hans Egli, who gave me many useful hints, and Dieter Kilsch, who took great care in reading the manuscript.

1 Sudoku Patterns and Sudokus

1.1 Definitions

Definition 1 (Sudoku Pattern) *>>pattern<<*

For $n = b^2$ ($b = 1, 2, 3, \ldots$), let P be an $(n \times n)$-matrix. Then P can be partitioned into n $(b \times b)$-matrices called *boxes* (or *subgrids*).

A *sudoku pattern of size* $n \times n$ is an $(n \times n)$-matrix the elements of which are any of the digits $0, 1, \ldots, n$, satisfying the following 3 conditions (the *B-R-C-conditions*):

(B) In each box (or subgrid), none of the digits $1, 2, \ldots, n$ appears more than once (whereas the digit 0 may appear arbitrarily often)

(R) In each row, none of the digits $1, 2, \ldots, n$ appears more than once (whereas the digit 0 may appear arbitrarily often).

(C) In each column, none of the digits $1, 2, \ldots, n$ appears more than once (whereas the digit 0 may appear arbitrarily often).

Definition 2 (Completion, extension, sudoku) *>>coexsu<<*

(i) A sudoku pattern is called *complete* if 0 does not occur in it.

(ii) For patterns P and Q, Q is called *an extension of* P if the two patterns coincide in all cells of P with positive values.

(iii) A sudoku pattern P is called a *sudoku* if there exists exactly one pattern Q which is complete and an extension of P.

1.2 Remarks

- In this script, the zeros are omitted in order to increase readability.
- From these definitions, it follows that every complete sudoku pattern is a sudoku.
- Of course, the digits $0, 1, 2, \ldots, n$ can be replaced by any collection of $n+1$ distinct signs, i.e. by a blank for 0 and some of the letters $\mathrm{A}, \mathrm{B}, \ldots$.

Example 1.1 (Size 1×1**)**

There are exactly two sudoku patterns, and exactly two sudokus, namely $\boxed{}$ and $\boxed{1}$.

Example 1.2 (Size 4×4**)**

The following two matrices (zeros omitted) are both sudoku patterns, but only the one to the right is a sudoku:

sdk 1 sdk4_00

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

sdk 2 sdk4_06

Example 1.3 (Size 9×9**)**

The following two matrices (zeros omitted) are both sudoku patterns, but only the one to the right is a sudoku:

sdk 3 sdk9_uo_empty

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

sdk 4 sdk9_uo_full

1.3 All sudokus of size 4×4

We follow Bertram Felgenhauer und Frazer Jarvis [1]:

> There are exactly 288 complete sudokus of size 4×4.

Proof: We suppose the first (upper left) box to be filled as shown below:

1	2		
3	4		

sdk 5 sdk4_01

Then in box 2 (upper right) the first row must contain 3 and 4, and the second row must contain 1 and 2. Therefore, there remain 4 possible cases:

1	**2**	**3**	**4**
3	**4**	**1**	**2**

sdk 6 sdk4_02

1	**2**	**3**	**4**
3	**4**	**2**	**1**

sdk 7 sdk4_03

1	**2**	**4**	**3**
3	**4**	**1**	**2**

sdk 8 sdk4_04

1	**2**	**4**	**3**
3	**4**	**2**	**1**

sdk 9 sdk4_05

Case I: Let's consider box 3 (lower left). The first column must be formed from of 2 and 4, and the second from 1 and 3. In each of the two columns, the digits can be permuted independently, each state of box 3 uniquely determining box 4. Therefore, there are 4 possibilities.

Case II: In box 3, the rows $(2, 3)$ and $(4, 1)$ lead to a contradiction (i.e. there would be no complete extension). There remain, for rows 3 and 4 of the full pattern, just the two possibilities $(2, 1, 4, 3)$ and $(4, 3, 1, 2)$. Interchanging these two rows gives rise to 2 possibilities.

Case III: This case is quite analogous to case II. There are 2 possibilities.

Case IV: This case is analogous to case I. There are 4 possibilities.

So the total number of possibilities amounts to $4 + 2 + 2 + 4 = 12$. All other possibilities can be obtained by a permutation of the digits $1, 2, 3, 4$. The total number of permutations is $4! = 24$. Therefore, there are

$$4! \cdot (4 + 2 + 2 + 4) = 288$$

complete 4×4 sudokus. Q.E.D.

To be a sudoku, a 4×4 pattern must have at least 4 clues. Any pattern with less than 4 clues has either none or more than one completion. The number of these *minimal* sudokus with just 4 clues is 25 728. Here are four of them:

1			
	4	**2**	
		4	

sdk 10

sdk4_004

		2	
2			**3**
4			

sdk 11

sdk4_005

			4
	4		
		1	
2			

sdk 12

sdk4_006

	2		
			4
	3		**1**

sdk 13

sdk4_007

Of course, the number 25 728 may be considerably reduced if we only count sudokus which are not obviously equivalent, especially by permuting the digits 1, 2, 3, 4.

The number of completions of a sudoku pattern with exactly 4 clues is always one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 18.

1.4 About the sudokus of size 9×9

How many complete sudokus are there of size 9×9? Independently from each other, Bertram Felgenhauer and Frazer Jarvis [1] have found this number to be

$$6670\,903752\,021072\,936690 \approx 6.671 \cdot 10^{21}.$$

If you consider sudokus equivalent if they can be mapped to each other by a permutation of the digits 1 through 9, the number of distinct complete sudokus still amounts to $18383\,222420\,692992 \approx 1.838 \cdot 10^{16}$.

Example 1.4 (Minimal sudoku) *>>minimal<<*

The next sudoku pattern is a sudoku, i.e., it has exactly one completion.

	1							**9**
			3			**8**		
						6		
				1	**2**	**4**		
7		**3**						
5								
8			**6**					
				4			**2**	
			7				**5**	

sdk 14 sdk9_17_1

It has only 17 clues (given digits). In 2012, Gary McGuire, Bastian Tugemann, and Gilles Civario published a computer-assisted proof that there are no sudokus with only 16 or less clues. For details and much background to the minimal-clues problem, see McGuire, Tugemann, Civario[4]. There is strong evidence that the number of 17-clue sudokus lies close to 50 000. Most of these were found by Gordon Royle. They may be downloaded from Royle's homepage [5].

Example 1.5 (Sudoku patterns, but no sudokus)

Of the following three sudoku patterns, none is a sudoku. The one to the left has 85 distinct completions. The one in the middle is the common part (the intersection) of all completions. It has the same completions as the original pattern. The one to the right differs from the original one only in that there is a 2 in cell $(2, 6)$. It is therefore a sudoku pattern with no completion at all, as all completions have a 1 in cell $(2, 6)$. Sudoku patterns are no sudokus if and only if the have no completion at all, or more than one completion.

		3		**9**		**6**	**4**	
	1		**7**		**8**		**2**	
		9				**4**		
	8		**4**		**5**		**9**	
		5				**8**		
	6		**8**		**2**		**5**	
	4	**1**		**6**		**2**	**3**	

sdk 15 sdk9_viel85

		3		**9**	**1**	**6**	**4**	
	1		**7**		**8**		**2**	
		9		**8**		**4**		**5**
	8		**4**		**5**		**9**	
4		**5**				**8**		
	6	**7**	**8**		**2**		**5**	
	4	**1**		**6**		**2**	**3**	

sdk 16 sdk9_viel85_Clos

		3		**9**	**2**	**6**	**4**	
	1		**7**		**8**		**2**	
		9		**8**		**4**		**5**
	8		**4**		**5**		**9**	
4		**5**				**8**		
	6	**7**	**8**		**2**		**5**	
	4	**1**		**6**		**2**	**3**	

sdk 17 sdk9_viel85_None

1.5 Completion by trial and error

Every sudoku can be completed, in theory, without any rules, just by trial and error. Take an empty cell and substitute in turn every digit which does not occur in the same row, or column, or box. Iterate the procedure until all the extensions but one cease to be sudoku patterns because they violate the B-R-C-conditions. The surviving one is the desired completion. This method is, of course, extremely laborious (and boring). The following rules make sudoku completion more exciting.

1.6 Constraint propagation

The aim of this paper is to present the *propagation rules* of sudoku completion as strictly, and as briefly as possible. We stick to the notions and denotations as used in Fowler[2]. The propagation rules are:

F Uniqueness of a digit in a given cell (F: "Field")

N Uniqueness of a cell for a given digit in a given box/row/column (N: "only")

B Box - row / Box - column interactions

T Use of naked and hidden tuples (T: "Tupel")

X X-chains (one-candidate chains)

Y Y-chains (pair chains)

W W-patterns (x-wing, swordfisch)

Sudokus which can be completed solely by these rules are called *constrained* in Fowler[2], and *unconstrained* otherwise. Sudoku puzzles published in books and newspapers are always supposed to be sudokus, not only sudoku patterns. It has to be remarked, however, that the rules of constraint propagation are in no way restricted to sudokus; they naturally apply to sudoku patterns. In section 9, we present an example in which the known uniqueness of the completion can be used to avoid guessing. The rules of constraint propagation can be applied to sudokus of any size. However, we shall now focus on size 9×9.

2 Elementary Sudokus (Rules F and N)

2.1 The elementary rules

Definition 3 (Digits barred from cells) *>><<*

We say that a given digit is *barred from a given empty cell* if the digit already occupies a cell in the same box, or the same row, or the same column.

Rule 1 (F: Field) *If from a given cell (field) all digits are barred but one, put this digit into the cell.*

Rule 2 (N: oNly cell) *There are three subrules:*

N_B **Box scanning for a digit** *If in a given box, a digit is barred from all empty cells but one, put the digit into this cell.*

N_R **Row scanning for a digit** *If in a given row, a digit is barred from all empty cells but one, put the digit into this cell.*

N_R **Column scanning for a digit** *If in a given column, a digit is barred from all empty cells but one, put the digit into this cell.*

We might be tempted to put it shorter, and just say, for example: If in a given box, a given digit is possible in just one cell, put the digit there. But this would be a misleading rule. In any sudoku (but not in any sudoku *pattern*), there is just one digit possible in any cell.

Definition 4 (Elementary rules, elementary sudokus) *>><<*

By the *elementary rules*, we understand the rules F, N_B, N_R, and N_C. We call a sudoku *elementary* if it allows completion solely by the elementary rules.

2.2 Independence of F, N_b, N_r, and N_c

The four rules are independent from each other in the sense that none of them can be replaced with a combination of the three others. We give four sudoku patterns each one of which can be extended by one, and only one, of the four rules.

Example 2.1 (Only F)

	2	3		9		6	4	
	1		7		8		2	
		9		8		4		5
	8		4		5		9	
4		5				8		
	6		8		2		5	
	4	1		6		2	3	

sdk 18 sdk9_uo_F1

$= F \Rightarrow$

	2	3		9	1	6	4	
	1		7		8		2	
		9		8		4		5
	8		4		5		9	
4		5				8		
	6	7	8		2		5	
	4	1		6		2	3	

sdk 19 sdk9_uo_F2

- For cell (2,6), there is only one possible digit, namely 1.
- For cell (7,3), there is only one possible digit, namely 7.

Example 2.2 (Only N_B)

	1	2	8	4				
	9	3				4		
	4				3			
1				3	4			
	5					1	3	4
			1	7				9
			6				2	
		1	3			7	4	
			4		5	8	1	

sdk 20 sdk9_uo_NB1

$= N_B \Rightarrow$

	1	2	8	4				
	9	3				4		
	4				3			
1			5	3	4			
	5					1	3	4
			1	7				9
			6		7	9	2	
		1	3			7	4	
			4		5	8	1	

sdk 21 sdk9_uo_NB2

- In box $B_{2,2}$ (center), digit 5 can only be placed in cell (4,4).
- In box $B_{3,2}$, digit 7 can only be placed in cell (7,6).
- In box $B_{3,3}$, digit 9 can only be placed in cell (7,7).

Example 2.3 (Only N_R)

				4	8	9		
	8	4	2			5	1	
	9		3			2	8	4
9						6	4	
	4	5				7		
	3	8			4	1		2
4	6				1	8	2	
8	2		4		5	3		
		9	8	2		4		

sdk 22 sdk9_uo_NR1

$= N_R \Rightarrow$

				4	8	9		
	8	4	2			5	1	
	9		3			2	8	4
9						6	4	
	4	5				7		
	3	8			4	1		2
4	6				1	8	2	5
8	2		4		5	3		
		9	8	2		4		

sdk 23 sdk9_uo_NR2

- In row 7, digit 5 is only possible in the last column.

Example 2.4 (Only N_C)

					1	**2**		**3**
	4			**5**			**7**	
								6
1					**7**			
	8						**9**	
			3					**2**
5		**8**					**2**	
				9			**8**	
3		**2**	**6**					

sdk 24 sdk9_uo_NC1

$= N_C \Rightarrow$

					1	**2**		**3**
	4			**5**			**7**	
								6
1					**7**		**3**	
	8						**9**	
			3					**2**
5		**8**					**2**	
				9			**8**	
3		**2**	**6**					

sdk 25 sdk9_uo_NC2

- In column 8, digit 3 can only be placed in the 4th row.

2.3 Most easy to complete (rule N_b)

If a sudoku is completed with the aid of a candidate list, then the most convenient rule is of course F. However, if the sudoku is completed "on sight", i.e., without any auxiliary notes, the easiest way to extend sudoku patterns is by rule N_B. Here is an example:

8→(4,2)	6→(8,4)	5→(2,5)	1→(5,9)
7→(5,6)	5→(8,9)	3→(2,9)	4→(2,4)
2→(8,5)	3→(9,8)	6→(4,3)	9→(3,8)
5→(5,2)	1→(6,2)	2→(5,8)	1→(4,5)
3→(5,4)	2→(6,3)	4→(8,7)	9→(6,5)
2→(7,2)	3→(6,7)	3→(1,2)	4→(1,8)
8→(8,1)	5→(7,4)	8→(1,5)	9→(2,6)
5→(4,8)	1→(7,8)	1→(2,7)	4→(3,2)
8→(7,6)	7→(8,8)	6→(6,8)	6→(2,2)
9→(8,2)	7→(9,2)	8→(2,8)	6→(3,6)
3→(8,6)	4→(9,5)	1→(3,4)	
9→(5,1)	2→(2,1)	9→(4,7)	
1→(8,3)	7→(2,3)	6→(5,5)	

1		9	7		2	5		6
5		8		3		7		2
3			2		4			7
		4				8		
7			8		5			4
4		3		7		6		9
6		5	9		1	2		8

sdk 26 sdk9_20min_240706LEICHT

The protocol on the left shows how to proceed.

2.4 Iteration of the elementary rules

Iterating the four elementary rules (ER) does not, in general, lead to a completion of a given sudoku. Some examples:

Example 2.5 (No elementary completion)

Iteration of the four basic methods stops when 31 digits are determined. None of the four methods can add more digits.

3	4		6					
		7						
	2			8		5	7	
					5			
	7			1			2	
			4					
	3	6		2			1	
						9		
					7		8	2

sdk 27 sdk9_NZZaS_230706

= (ER) ⇒

3	4	5	6	7	1	2	9	8
		7	2	5				
	2			8		5	7	
			7		5			
	7			1			2	
			4		2			
	3	6		2			1	
						9		
					7		8	2

sdk 28 sdk9_NZZaS_230706e

Example 2.6 (Elementary completion)

			4	9		2		7
	4		1			9		
				7	2			1
	8					4		2
7								5
6		2					1	
4			2	6				
		8			5		6	
5		3		8	7			

sdk 29 sdk9_tbz_060606

= (ER) ⇒

8	5	1	4	9	6	2	3	7
2	4	7	1	3	8	9	5	6
9	3	6	5	7	2	8	4	1
3	8	5	6	1	9	4	7	2
7	1	4	8	2	3	6	9	5
6	9	2	7	5	4	3	1	8
4	7	9	2	6	1	5	8	3
1	2	8	3	4	5	7	6	9
5	6	3	9	8	7	1	2	4

sdk 30 sdk9_tbz_060606e

In this example, completion can even be attained by an iteration of rule N_B.

2.5 Problems

The following three sudokus can be completed by any one of the four elementary rules:

8		5		9		7		2
3		4	2		7	6		5
		9	7		1	2		
1								8
		2	8		5	1		
5		6	3		4	8		7
9		1		2		3		6

sdk 31 sdk9_20min_240706L_trsf

6								3
			7	6	3			
		5				8		
	3			5			4	
	4	2	8		1	5	9	
	6			2			7	
		3		9		6		
			1	7	4			
4								1

sdk 32

sdk9_BaA_071209K_Z15_trsf

4	3		5		7		2	9
5								7
			1	2	9			
2		3				5		4
		9				2		
1		7				6		3
			3	5	6			
3								8
9	1		2		4		3	6

sdk 33 sdk9_heute_170806E_trsf

The following 9 sudokus can be completed by rule N_B only:

		2		1				6
				9				3
	8				4	9		
	4		7					
7			9		1			8
					3		6	
		3	4				9	
1				7	8			
6				2		4		

sdk 34 sdk9_BaA_120410_Z24_trsf

6								3
			7	6	3			
		5		4		8		
	3			5			4	
	4	2	8		1	5	9	
	6			2			7	
		3		9				
			1	7	4			
4								1

sdk 35

sdk9_BaA_071209K_Z99_trsf

9		3				5		6
			3		1			
5								2
	4			9			8	
			4		2			
	8			6			1	
7								4
			5		7			
1		2				8		5

sdk 36 sdk9_heute_160806E

1			2	9				7
					6			
	3		7				5	
	1			4		7		3
2								6
6		4		5			8	
	4				1		7	
			3					
9				6	4			2

sdk 37 sdk9_heute_100706

1			8			9		
	4			1			6	
		2			5			7
8			3			7		
	1			2			5	
		9			7			4
6			5			1		
	9			6			4	
		1			2			8

sdk 38 sdk9_heute_220606

9								5
	8		6		7		2	
			9		2			
	4	8		9		7	1	
			7		5			
	3	7		1		5	8	
			1		9			
	9		5		4		7	
1								2

sdk 39 sdk9_heute_240806

	5		3	8		9	4	
4			6		1	5		8
7	8							
	2						3	7
5								9
8	3						6	
							5	1
3		1	2		4			6
	6	5		1	3		2	

sdk 40 sdk9_heute_210806

6		5	1		8	3		2
9		2				7		5
3				7				8
			6		1			
7				5				1
5		3				6		7
2		1	7		3	5		4

sdk 41 sdk9_heute_240706

			4			5		
		2		5				
	7		2			8		9
4		6		3				
	3		1		9		8	
				8		3		4
3		5			1		2	
				7		1		
		9			3			

sdk 42 sdk9_heute_250706

The following three sudokus can be completed by rule N_R alone:

5				1				8
			2	6	9			
		6				9		
	9			5			1	
7	1		8		4		3	5
	6			7			2	
		1				4		
			4	2	1			
9				3				6

sdk 43 sdk9_BaA_071209K_trsf

	5			4			9	
7		1						2
	8				3			
			3		7	5		
8								6
		6	9		8			
			7				2	
5						9		3
	6			1			7	

sdk 44 sdk9_NEWS_251109

1			9		7	5		4
3		4	8			9		
9				4		3		2
7		5		6				1
		9			2	1		6
4		8	7		3			9

sdk 45 sdk9_tbz_140806

The following three sudokus can be completed by rule N_C alone:

8	9							5
				6		7		4
	5				1			
		8	6		4			
	1			5			6	
			2			1		
			7				8	
5		6		3				
9							2	7

sdk 46

sdk9_20min_230806M_Z66_trsf

			8					
9					4			7
3	5				2		4	9
	3	8						5
				7				
4						2	6	
2	1		5				7	3
7			2					4
					6			

sdk 47 sdk9_NEWS_020909

	5							
7	1	9		4				
	8				7			2
	9			3				6
	2	1	9		4	3	8	
5				6			9	
8			7				4	
				8		7	5	9
							3	

sdk 48 sdk9_tbz_030706

The following three sudokus can be completed by rule F alone:

3		1	2	5		8		4
		6	8			5		
		9				6		8
2								7
8		5				1		
		8	4		2	9		
7		3		8	5	4		2

sdk 49

sdk9_20min_270809M_Z16_trsf

		9	1		6	7		
				8				
3								2
2			7		3			8
	5			4			9	
1			2		8			4
4								7
				6				
		3	4		5	1		

sdk 50 sdk9_BaA_141209K

	2							
		7	8		6	1		2
				2			3	
	9			3			6	
		4				7		
	3			1			4	
	1			4			7	
6		2	3		9	8		
							5	

sdk 51

sdk9_20min_141209M_Z12_trsf

The following six sudokus need for completion at least three of the elementary rules, the last two require all four:

		5						4
		7			5	2		
3	2			9	7		5	
							3	
		1		2		9		
	5	9						
	9		6	7			4	2
		2	8			1		
5						6		

sdk 52

sdk9_BaA_141009M_Z49_trsf

9								6
		7	5	3	4			
							5	
							4	
	1		9	8	2		6	
	3						8	
	4							
			7	4	1	2		
8								7

sdk 53 sdk9_ta_290410_Z42_trsf

			7			2	5	
		8						6
	9			3				8
5				2				
		3	9		4	8		
				6	7			1
7				1			4	
3						6		
	1	6			2			

sdk 54

sdk9_NEWS_261009_Z44_trsf

		8			6		2	
		6	9					4
1	4			7				
	2							1
		5				3		
4							8	
				6		7	3	5
3					5	4		
	9		1			8		

sdk 55 sdk9_BaA_250809_Z11_trsf

		9					4	6
				3	7	8		
					3	1	2	
				8				
	5	4	6					
		8	4	1				
2	9					7		

sdk 56 sdk9_NZZaS_280609_trsf

	3						5	7
				3	7	9	6	
4	7				1	2		
5	2				8			
		4			3			5
6		3	1					
8							9	
	4			9	5			1
7							4	

sdk 57 sdk9_SpP_135_trsf

3 Candidate Tables

For a given sudoku, we obtain a *candidate table* by writing down, in each empty cell, all digits which do not occur in the same box, or the same row, or the same column. Completing a sudoku means reducing the candidate table up to the point where to each cell just one candidate is attributed.

3.1 Candidate tables and elementary rules

Although elementary sudokus can comfortably be completed without auxiliary notes, the elementary rules can be illustrated with candidate tables.

1 2 3 4 5 7 8	1 5 7 8	1 2 3 4 7	4 5 7 9	3 4 6	3 6 7 9	1 3 4 6 8 9	4 5 6 9	1 3 4 5 6 8 9
3 4 5 7	**9**	3 4 7	**8**	3 4 6	**1**	3 4 6	**2**	3 4 5 6
1 3 4 5 8	1 5 8	**6**	4 5 9	**2**	3 9	**7**	4 5 9	1 3 4 5 8 9
1 4 6 7	**3**	1 4 7	1 2 7	**9**	2 7	1 4 6	**8**	1 4 5 6 7
1 7 8 9	1 7 8	**5**	**6**	1 3 8	**4**	**2**	7 9	1 7 9
1 4 6 7 8 9	**2**	1 4 7 9	1 7	**5**	7 8	1 4 6 9	**3**	1 4 6 7 9
1 2 3 6 9	1 6	**8**	1 2 4 9	**7**	2 6 9	**5**	4 6 9	2 3 4 6 9
2 6 7 9	**4**	2 7 9	**3**	6 8	**5**	6 8 9	**1**	2 6 7 8 9
1 2 3 5 6 7 9	1 5 6 7	1 2 3 7 9	1 2 4 9	1 4 6 8	2 6 8 9	3 4 6 8 9	4 6 7 9	2 3 4 6 7 8 9

sdk 58 sdk9_heute_170806M

1 2 6	**5**	2 6	**3**	**8**	2 7	**9**	**4**	2
4	9	2 3 9	**6**	2 7 9	**1**	**5**	7	**8**
7	**8**	2 3 6 9	4 5 9	2 4 5 9	2 5 9	1 2 3 6	1	2 3
1 6 9	**2**	4 6 9	1 4 5 8 9	4 5 6 9	5 6 8 9	1 4 8	**3**	**7**
5	1 4 7	4 6 7	1 4 7 8	2 3 4 6 7	2 6 7 8	1 2 4 8	1 8	**9**
8	**3**	4 7 9	1 4 5 7 9	2 4 5 7 9	2 5 7 9	1 2 4	**6**	2 4 5
2 9	4 7 9	2 4 7 8 9	7 8 9	6 7 9	6 7 8 9	3 4 7 8	**5**	**1**
3	7 9	**1**	**2**	5 7 9	**4**	7 8	7 8 9	**6**
9	**6**	**5**	7 8 9	**1**	**3**	4 7 8	**2**	4

sdk 59 sdk9_heute_210806

Rule F In the sudoku on the left, there is no cell with a single candidate. Therefore, this rule has no effect. In the sudoku on the right, however, there are 6 such cells. Therefore, rule F allows us to immediately expand the sudoku by 6 more digits.

Rule N_B In the sudoku on the left, this rule leads to two more final digits: 3 in box $B_{2,2}$, 5 in box $B_{2,3}$. In the sudoku at the right, this rule leads to 9 more final digits: 1 in box $B_{1,1}$, 6 and 7 in box $B_{1,3}$, 3 in box $B_{2,2}$, 5 in box $B_{2,3}$, 8 in box $B_{3,1}$, 5 in box $B_{3,2}$, 3 and 9 in box $B_{3,3}$.

Rule N_R In the sudoku on the left, this rule leads to two final digits: 5 in cell $(4,9)$, and 3 in cell $(5,5)$. In the sudoku at the right, this rule leads to 6 more final digits: 1 in cell $(1,1)$, 7 in cell $(1,6)$, 3 in cell $(2,3)$, 3 in cells $(5,5)$ and $(7,7)$, and 5 in cell $(8,5)$.

Rule N_C In the sudoku on the left, this rule leads has no effect. In the sudoku at the right, it leads to to 7 more final digits: 1 in cell $(5,2)$, 8 in cell $(7,3)$, 3 in cell $(5,5)$, 6 in cell $(3,7)$, 9 in cell $(8,8)$, 3 in cell $(3,9)$, 5 in cell $(6,9)$, The rules may overlap. In the sudoku on the right, for instance, candidate 3 in cell $(5,5)$ is the final digit by rule N_B, as well as by rules N_R and N_C (but not by F).

If we apply, for instance, rule N_B to both sudokus, we get

	9		**8**		**1**		**2**	
		6		**2**		**7**		
	3			**9**			**8**	**5**
		5	**6**	**3**	**4**	**2**		
	2			**5**			**3**	
		8		**7**		**5**		
	4		**3**		**5**		**1**	

sdk 58

1	**5**		**3**	**8**		**9**	**4**	
4			**6**		**1**	**5**	**7**	**8**
7	**8**					**6**		
	2						**3**	**7**
5				**3**				**9**
8	**3**						**6**	**5**
		8				**3**	**5**	**1**
3		**1**	**2**	**5**	**4**		**9**	**6**
	6	**5**		**1**	**3**		**2**	

sdk 59

3.2 Another look at candidate tables

As candidate tables shrink in the process of constraint propagation, the candidates of an empty field may vary although the sudoku does not. We call a sudoku *compatible* with a candidate table if for every empty field of the sudoku, the corresponding field of the candidate table contains, possibly among others, the clue of the completed sudoku. Therefore, we get a candidate table which is compatible with a given sudoku by filling in, for every empty field, all the candidates $1, ..., 9$.

As remarked at the beginning, the 7 rules of constraint propagation do not really presuppose sudokus. They apply equally well to sudoku patterns. Then there are three possibilities:

1. Constraint propagation ends with one or more cells containing no candidates at all. The sudoku pattern has no completion.

2. Constraint propagation ends with each cell containing exactly one candidate. Then the sudoku pattern is a sudoku, and the candidates are the final digits of the solution.

3. Constraint propagation stabilizes with one ore more cells containing more than one candidate. Then the given sudoku pattern may have more than one solution. Or it has just one solution, which has to be found by other methods, and therefore is a sudoku.

Now a candidate table is *compatible* with a given sudoku *pattern* if for every empty field of the pattern, the corresponding field of the candidate table contains, possibly among others, the clues of *all* completions.

4 Box-Row and Box-Column Interactions

`>>brcint<<`

4.1 Rule B and its 4 subrules

Rule $B \succ R$ If within some box, the candidates of a given digit are restricted to one single row, all further candidates of the digit that occur within this row but outside the given box can be eliminated.

Rule $B \succ C$ If within some box, the candidates of a given digit are restricted to one single column, all further candidates of the digit that occur within this column but outside the given box can be eliminated.

Rule $R \succ B$ If within some row, the candidates of a given digit are restricted to one single box, then they can be eliminated from the other two rows of the box.

Rule $C \succ B$ If within some column, the candidates of a given digit are restricted to one single box, then they can be eliminated from the other two columns of the box.

The following examples show sudokus which can be completed almost by elementary rules alone. Just once in each case, one of the B rules is necessary.

Example 4.1 (Rule $B \succ R$) *>>noauxiliarynotes<<*

	9				**8**	**1**	**6**	
		1	**7**			**9**	**2**	
2								**3**
3		**8**	**4**		**1**	**5**		**6**
	4	**9**	**8**			**2**	**3**	
6					**7**	**8**		**4**
8			**6**	**4**				**9**
				7	**2**	**6**		
	6	**7**	**1**	**8**			**5**	**2**

sdk 60 sdk9_gch_1

4 5 7	**9**	3 4 5	2 3 5	2 3 5	**8**	**1**	**6**	5 7
4 5	3 5 8	**1**	**7**	3 5 6	3 4 5 6	**9**	**2**	5 8
2	5 7 8	**6**	5 9	**1**	~~4~~ 5 9	(4) 7	(4) 7 8	**3**
3	2 7	**8**	**4**	2 9	**1**	**5**	7 9	**6**
1 5 7	**4**	**9**	**8**	5 6	5 6	**2**	**3**	1 7
6	1 2 5	2 5	2 3 5 9	2 3 5 9	**7**	**8**	1 9	**4**
8	1 2 3 5	2 3 5	**6**	**4**	3 5	3 7	1 3 7	**9**
1 4 5 9	1 3 5	3 4 5	3 5 9	**7**	**2**	**6**	1 4 8	1 8
4 9	**6**	**7**	**1**	**8**	3 9	3 4	**5**	**2**

sdk 61 sdk9_gch_1_e

By applying rule N_B twice, the sudoku on the left is extended to the sudoku on the right. Then in box $B_{1,3}$, candidate 4 is restricted to row 3. Therefore by rule $B \succ R$, candidate 4 can be eliminated in cell $(3, 6)$. As a consequence, 4 is put into cell $(2, 6)$ by rule N_B. Then completion can be achieved by rule F alone.

Example 4.2 (Rule $R \succ B$)

			5				2	
1	8			4				
		5					6	
		6	1					4
		2		3		8		
3					7	9		
	5					7		
				2			1	5
	7				4			

sdk 62 sdk9_ta_170809

4 6 9	3 6	3 4 7 9	**5**	1 7 9	1 3 8	1 3 4	**2**	3 7 8
1	**8**	3 7	**6**	**4**	**2**	**5**	**9**	3 7
4 9	**2**	**5**	3 8 9	1 7 9	1 3 8	1 3 4	**6**	3 7 8
7	**9**	**6**	**1**	**8**	**5**	**2**	**3**	**4**
5	**1**	**2**	**4**	**3**	**9**	**8**	**7**	**6**
3	**4**	**8**	**2**	**6**	**7**	**9**	**5**	**1**
2 6 8 9	**5**	3 9	3 8 9	1 9	1 3 6 8	**7**	**4**	2 3 9
4 6 8 9	3 6	3 4 9	**7**	**2**	3 6 8	3 6	**1**	**5**
2 6 9	**7**	**1**	3 9	**5**	**4**	3 6	**8**	2 3 9

sdk 63 sdk9_ta_170809_e

Again, the sudoku on the left is extended to the sudoku on the right by elementary rules alone. Then in row 8, candidate 9 is restricted to box $B_{3,1}$. By rule $R \succ B$, all other instances of candidate 9 can be eliminated in this box. Then completion can be achieved by rule F alone.

Example 4.3 (Rule $B \succ C$)

				3	1	6	4	
5					6	2	9	
6								
	1		7					
		4				1		
					8		6	
								9
	9	7	4					2
	2	5	3	7				

sdk 64 sdk9_ta_280708S

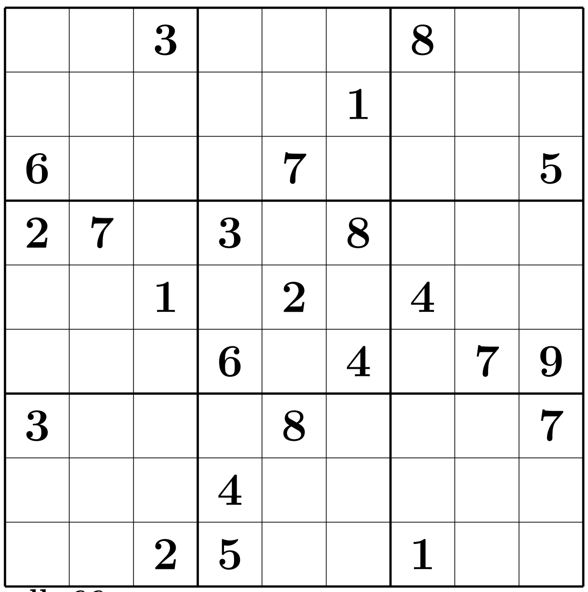
2 7 8 9	7 8	2 8 9	2 5 9	**3**	**1**	**6**	**4**	5 8
5	**3**	**1**	**8**	**4**	**6**	**2**	**9**	**7**
6	**4**	2 8 9	2 5 9	2 5 9	**7**	3 5 8	3 5 8	**1**
2 8 9	**1**	**6**	**7**	2 5 9	**4**	5 8 9	2 5 8	**3**
2 7 8 9	5 7 8	**4**	**6**	2 5 9	**3**	**1**	2 5 7 8	5 8
3	5 7	2 9	2 5 9	**1**	**8**	5 7 9	**6**	**4**
4	**6**	**3**	**1**	**8**	**2**	5 7	5 7	**9**
1 8	**9**	**7**	**4**	**6**	**5**	3 8	1 3 8	**2**
1 8	**2**	**5**	**3**	**7**	**9**	**4**	1 8	**6**

sdk 65 sdk9_ta_280708S_e

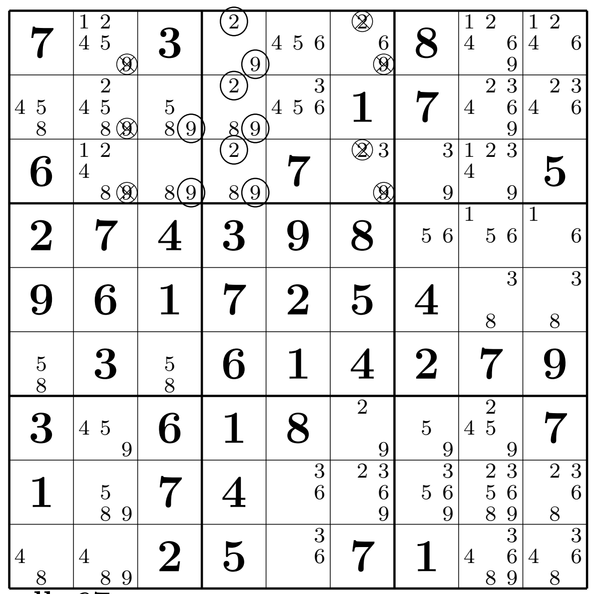
Again, the sudoku on the left is extended to the sudoku on the right by elementary rules alone. Then in box $B_{3,1}$, candidate 8 is restricted to column 1. By rule $B \succ C$, the other instances of candidate 8 can be eliminated in column 1. Then by rule N_B, cell $(5, 2)$ has to be set to 8. Completion can now be achieved by rules F and N_B.

In the first and the last of these three examples, candidate tables are not really needed. In many cases, rules $B \succ R$ and $B \succ C$ lead, together with an elementary rule, directly to another final digit.

Example 4.4 (Rule $C \succ B$)



sdk 66 sdk9_ta_270310_Z36_trsf

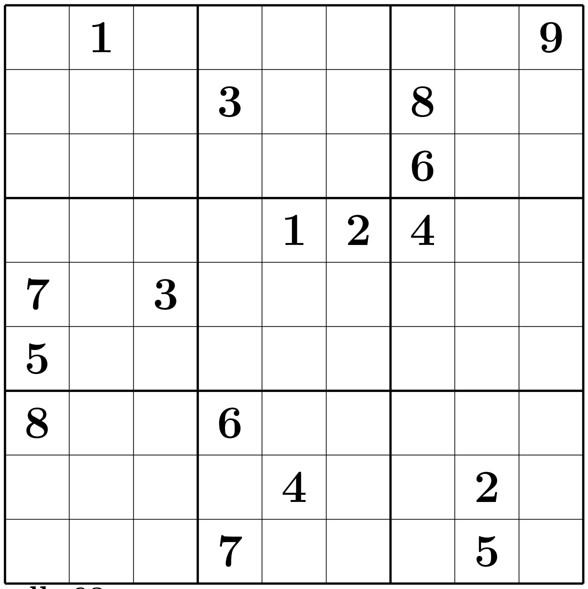


sdk 67 sdk9_ta_270310_Z36_trsf_e

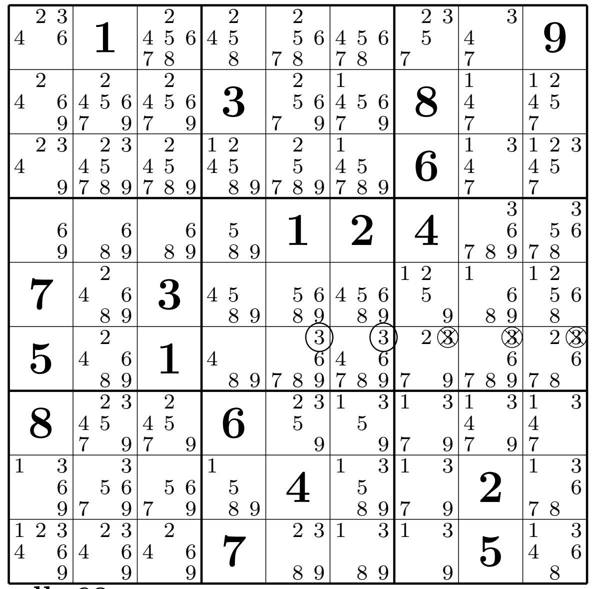
Again, the sudoku on the left is extended to the sudoku on the right by elementary rules. In column 3, candidate 9 is restricted to box $B_{1,1}$. Therefore by rule $C \succ B$, candidate 9 can be eliminated in the rest of this box. In column 4, candidates 2 and 9 are both restricted to box $B_{1,2}$ and can therefore be eliminated in cells $(1, 6)$ and $(3, 6)$.

Example 4.5

We return to the minimal sudoku with just 17 clues from example 1.4. It can be completed solely by the elementary rules $B \succ R$, and $B \succ C$.

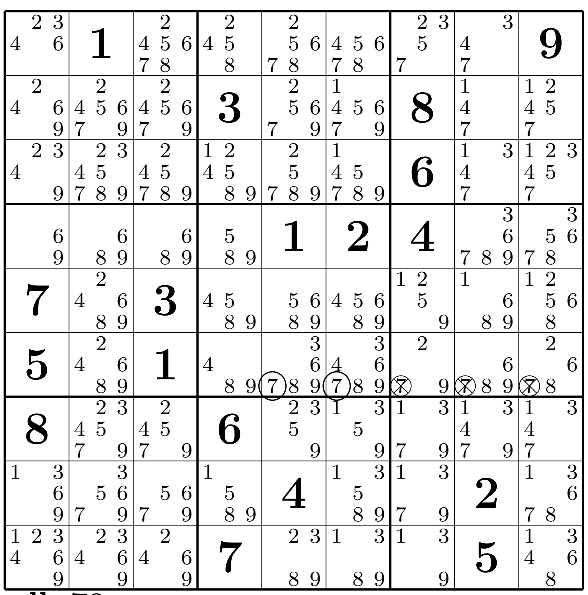


sdk 68 sdk9_17_1

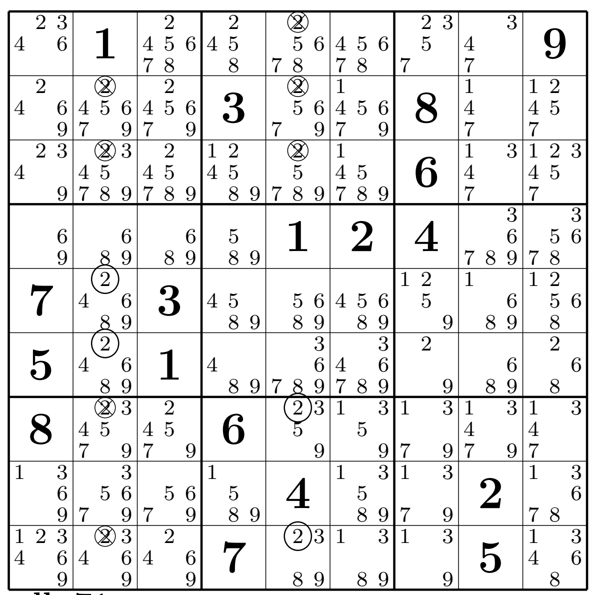


sdk 69 sdk9_17_1_XX0001

Rule N_B requires cell $(6,3)$ of sudoku 68 to be set to 1. The result is sudoku 69. Then in box $B_{2,2}$, candidate 3 is restricted to row 6. Therefore by rule $B \succ R$, candidate 3 can be eliminated from cells $(6,7)$, $(6,8)$, and $(6,9)$. This leads to sudoku 70.

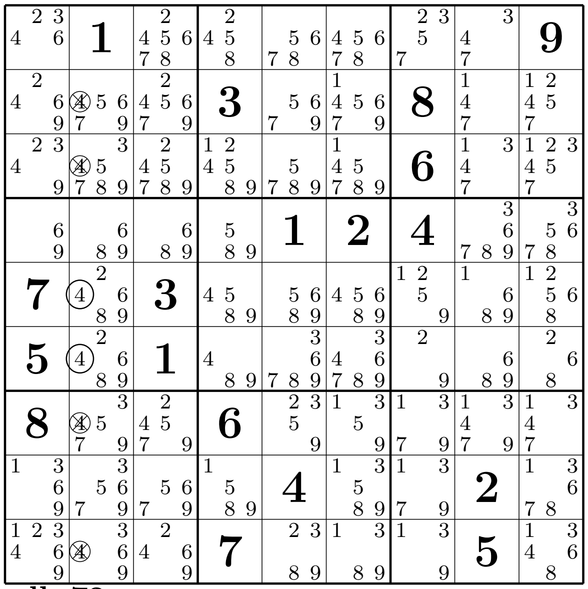


sdk 70 sdk9_17_1_XX0002

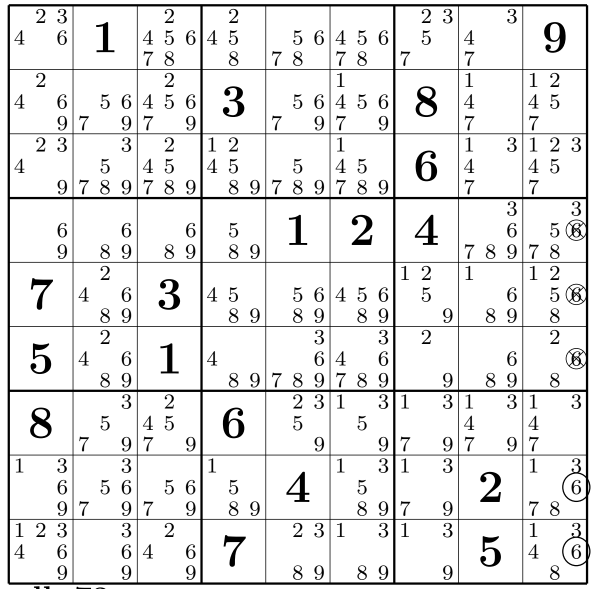


sdk 71 sdk9_17_1_XX0003

Also in box $B_{2,2}$, candidate 7 is restricted to row 6. Therefore by rule $B \succ R$, candidate 7 can be eliminated from cells $(6,7)$, $(6,8)$, and $(6,9)$. This leads to sudoku 71. Then rule $B \succ C$ can be applied to candidate 2 in column 2 as well as in column 5, and we get sudoku 72.

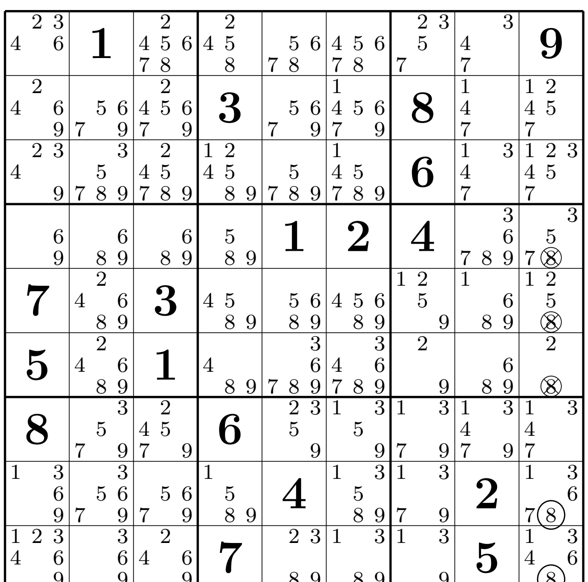


sdk 72 sdk9_17_1_XX0004

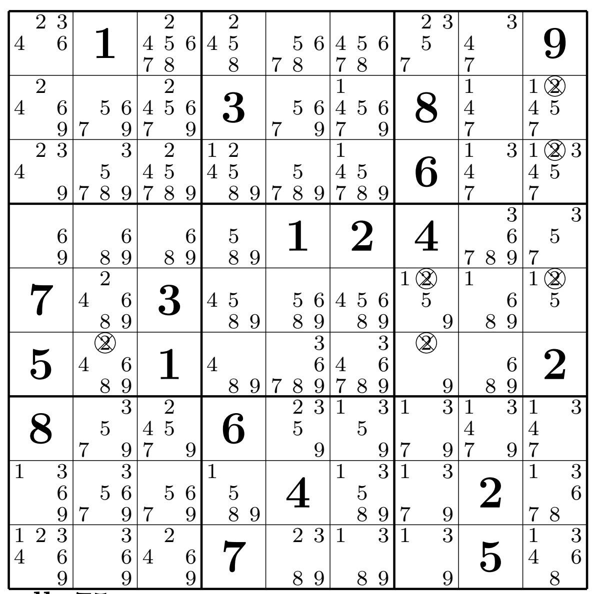


sdk 73 sdk9_17_1_XX0005

Rule $B \succ C$ can be applied to candidate 4 in column 2 of sudoku 72, and to candidate 6 in column 9 of sudoku 73.



sdk 74 sdk9_17_1_XX0006



sdk 75 sdk9_17_1_XX0007

So far, applying rules $B \succ R$ and $B \succ C$ has not resulted in any additional digit, but only in reducing the total number of candidates from 289 to 269. But when we now apply rule $B \succ C$ to candidate 8 in column 9 of sudoku 74, cell $(6, 9)$ has to be put to 2, and we get sudoku 75. Then by repeated applications of elementary rules (F and N_B suffice), we get sudoku 76.

3 4 6	**1**	4 5 6 7 8	4 5	5 6 7 8	4 5 6 7 8	**2**	3 4 7	**9**
2 4 6 9	5 6 7 9	2 4 5 6 7 9	**3**	5 6 7 9	1 4 5 6 7 9	**8**	1 4 7	1 4 5 7
3 4 9	3 5 7 8 9	4 5 7 8 9	**2**	5 7 8 9	1 4 5 7 8 9	**6**	1 3 4 7	1 3 4 5 7
6 9	6 8 9	6 8 9	5 9	**1**	**2**	**4**	3 7	3 ~~5~~ 7
7	**2**	**3**	4 ~~5~~ 9	~~5~~ 6 9	4 ~~5~~ 6 9	~~1~~ 5	**8**	1 ~~5~~
5	**4**	**1**	**8**	3 7	3 7	**9**	**6**	**2**
8	3 5 7	2 4 5 7	**6**	2 3 5	~~1~~ 3 5	1 3 7	**9**	1 3 4 7
~~1~~ 3 6 9	3 5 6 7 9	5 6 7 9	1 ~~5~~ ~~9~~	**4**	~~1~~ 3 5 8 9	~~1~~ 3 7	**2**	~~1~~ 3 6 7 8
1 2 3 4 6 9	3 6 9	2 4 6 9	**7**	2 3 8 9	~~1~~ 3 8 9	1 3	**5**	1 3 4 6 8

sdk 76 sdk9_17_1_XX0014

	1					**2**		**9**
			3			**8**		
			2			**6**		
				1	**2**	**4**		
7	**2**	**3**					**8**	
5	**4**	**1**	**8**			**9**	**6**	**2**
8			**6**				**9**	
				4			**2**	
			7				**5**	

sdk 77 sdk9_17_1_el

In sudoku 76, cells $(5, 7)$ and $(8, 4)$ have to be set to 5 and 1, respectively. Therefore, these candidates are already crossed out in the associated cells.

Remark Sudoku 76 is elementary and is obtained from sudoku 75 by elementary rules alone. But sudoku 75 itself is *not* elementary. At first, this seems to be a contradiction. The explanation lies in the fact that the candidate table of sudoku 75 has previously been reduced by non-elementary rules also. It comprises 265 candidates. However, if we started out with the 19 clues (definitely set digits) of sudoku 75 from scratch, we would get a total of 278 candidates in the table. Sudokus 76 and 77 have the same clues. The fact that sudoku 77 is elementary is mirrored by that its candidate table is equal to, and not an extension of, the candidate table of sudoku 76.

4.2 Problems

The next two sudokus can be completed by elementary rules and a single application of rule $B \succ R$:

4		3	6					7
		5	9		8			2
		4				3		6
3		8				1		
2		6	3		7	4		
7					5	8		1

sdk 78 sdk9_ta_200210_Z19_trsf

1			2		3			5
		5		9	8	3		
	5	2				7	8	
	6						5	
	9	3				2	1	
		8	9	7	4	5		
6			8		2			9

sdk 79 sdk9_20min_020909M_Z14_trsf

The next two sudokus can be completed by elementary rules and a single application of rule $B \succ C$:

4					3	2		7
3		5	4		8	6		
6		9				3		
		8				7		5
			3		1	4		8
7		2	6					1

sdk 80 sdk9_ta_200210_Z19_trsf_diag

8		9			2			5
	6			4	3			
						8		
				6		5		
6		8				4	2	7
		2		3				
		4						
			1	2			7	
9			5			1		6

sdk 81 sdk9_ta_021109_Z52_trsf

The next two sudokus can be completed by elementary rules and a single application of $R \succ B$, and $C \succ B$, respectively:

2	4						3	8
6								2
			2		4	7		
			4	2	9			
		2				6		
			3	5	6			
		9	6		5	2		
8								5
5	3						4	1

sdk 82 sdk9_Knaur_121_Z11_trsf

			6	7				
	8					2		5
	4				9			
		5	4					
	1			9			7	
					2	1		
			1				5	
6		7					4	
				8	3			2

sdk 83 sdk9_ta_170809_Z33_trsf_diag

The next 6 sudokus can be completed by elementary rules and repeated (less than 10) applications of rules B.

					9	7		
	4			2			3	
8			6					
1						4		
	5		8	4	3		6	
		7						
					6			5
	3			5			2	
		6	9					

sdk 84 sdk9_ta_230709_Z67_trsf

7					1		6	3
	3		9			8		
2								
4				2		7		
		9				1		
		2		7				6
								9
		7			6		4	
5	1		8					2

sdk 85 sdk9_tbz_100811_Z55_trsf

	1	**8**		**2**				
					9			**4**
		4		**3**		**6**		**5**
	3							
7				**5**		**9**		**3**
							1	
2		**3**		**4**		**8**		
5			**1**					
				7		**5**	**4**	

sdk 86 sdk9_tbz_270411_Z51_trsf

2		**7**				**3**		**5**
					4			
1								**4**
	7			**4**			**6**	
			9	**1**	**7**			
	5			**3**			**2**	
7								**2**
			8		**6**			
9		**3**				**8**		**7**

sdk 87 sdk9_ta_060210_Z24_trsf

		2				**9**		
			9		**5**			
	3		**2**		**6**		**8**	
	8	**4**				**6**	**3**	
	6	**5**				**8**	**9**	
	5		**6**		**9**		**2**	
			3		**8**			
		7				**4**		

sdk 88 sdk9_tbz_210710_Z55_trsf

		5						**4**
		9	**5**				**8**	
	1		**4**				**9**	**5**
	5	**2**			**7**			
				2				
			8			**6**	**7**	
3	**7**				**4**		**2**	
	8				**1**			
2						**3**		

sdk 89 sdk9_ta_291209_Z89_trsf

5 Tuple Reduction

5.1 Open tuples

Definition 5 (Associated cells) *>>assoc<<*

Box association We say that cells are *box-associated*, or *associated with respect to a box*, if they lie within one and the same box.

Row association We say that cells are *row-associated*, or *associated with respect to a row*, if they lie within one and the same row.

Column association We say that cells are *column-associated*, or *associated with respect to a column*, if they lie within one and the same column.

Associated cells We call cells *associated*, if they are contained in a common box, or a common row, or a common column.

In graph theory, cells would be called vertices, and associated cells correspond to adjacent vertices. Applied to sudokus, however, the term *adjacent cell* could obviously be misleading.

Suppose that some unoccupied cells are associated with respect to a box, or a row, or a column. Then the number of distinct candidates cannot be smaller than the number of cells, as otherwise, there would be no completion.

Definition 6 (Open tuple, irreducible) *>>otupel<<*

(i) If in a set of associated unoccupied cells the number of distinct candidates does not exceed the number of cells, we say that the cells form an *open tuple*. Open tuples are sometimes called *exact* or *naked* tuples.

(ii) By an *irreducible open tuple* we understand an open tuple which does not contain a proper subtuple of cells which is also open.

Technically speaking, an open tuple t is an ordered pair of two sets of equal size, the first consisting of candidates (digits from 1 through 9), the second of unoccupied associated cells (fields):

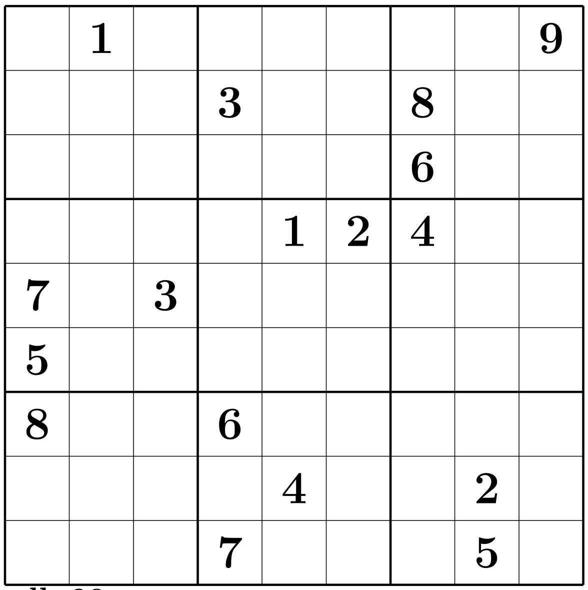
$$t = (\{c_1, \ldots, c_k\}, \{f_1, \ldots, f_k\}).$$

If the cells $f_1, \ldots, f_k$ are box-associated, we say that the candidates $c_1, \ldots, c_k$ form an open tuple with respect to the box in question, and analogously, if the cells are row- or column-associated.

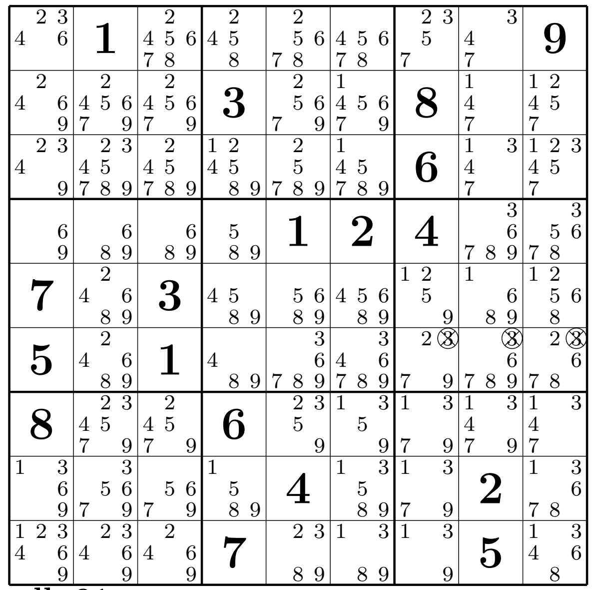
In the sudoku literature, open tuples of cardinality 2 are know as *open* (or *naked*) *pairs*, open tuples of cardinality 3 as *open triples*, etc. Open tuples of cardinality 1 we consequently call *open singletons*.

We again return to the minimal sudoku (example 1.4) with just 17 clues. As shown in section 4, it can be completed with rules F, N, and B. It can also completed with F, N, and T.

Example 5.1 (Minimal sudoku)

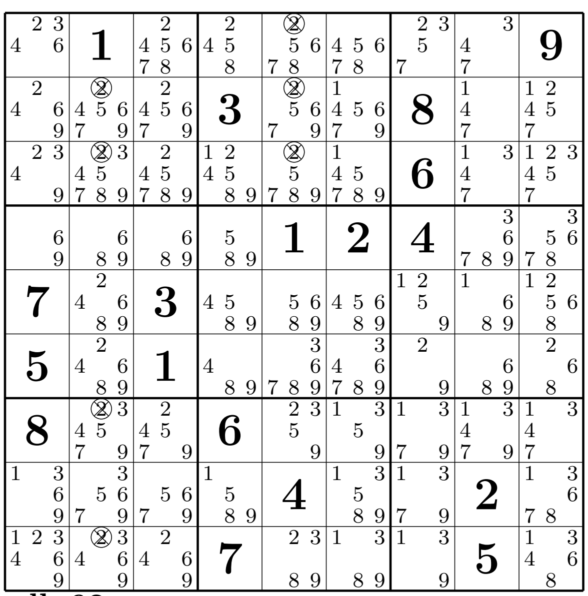


sdk 90 sdk9_17_1

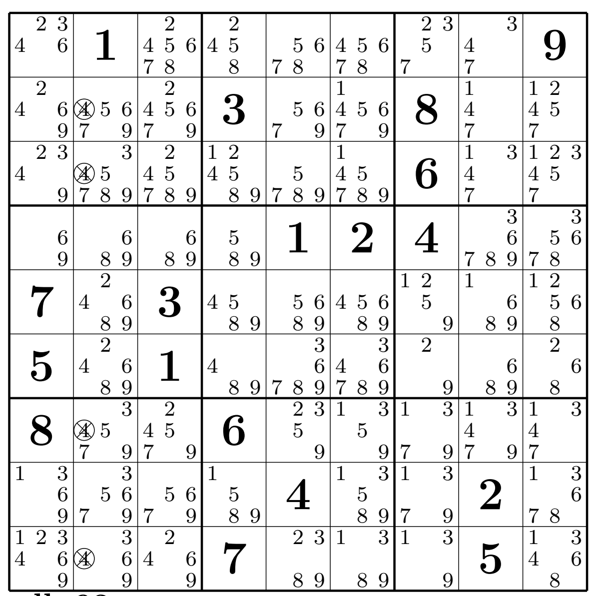


sdk 91 sdk9_17_1_XX0001

According to rule N_B, digit 1 has to be put into cell $(6, 3)$ of sudoku 90. Thus we get sudoku 91. The candidate table now contains, at the beginning of row 4, the open triple $\{6, 8, 9\}$. Therefore, these candidates can be eliminated in the remaining 6 cells of row 4, and we get sudoku 92

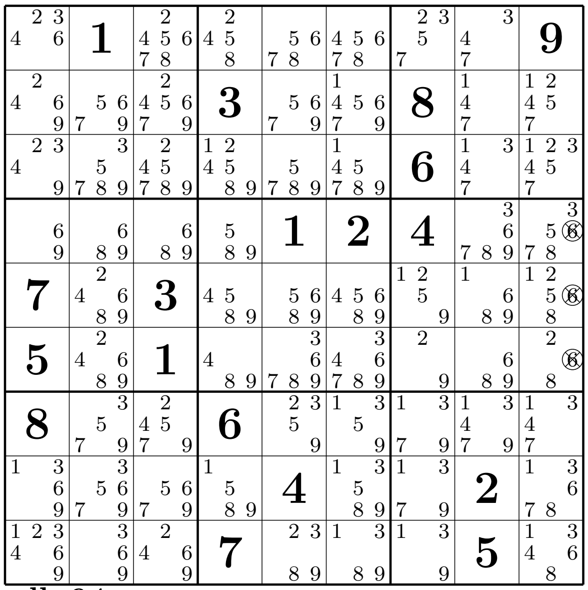


sdk 92 sdk9_17_1_XX0003

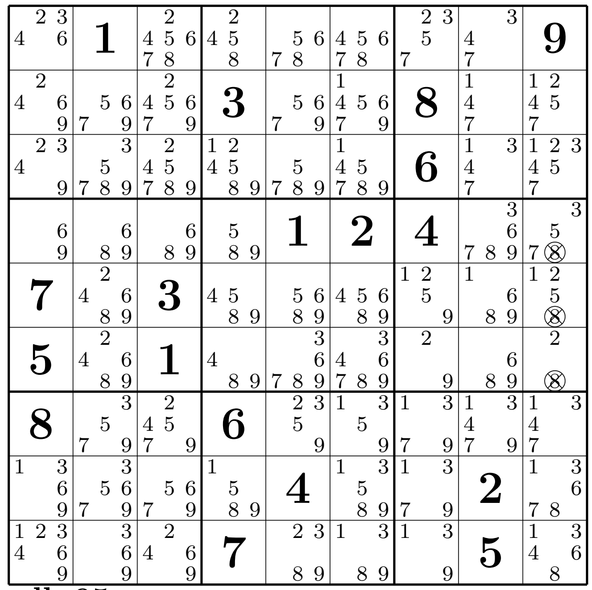


sdk 93 sdk9_17_1_XX0004

At the end of row 4 of sudoku 92, we now see the open pair $\{3, 7\}$, which allows us to eliminate these candidates in the remaining cells of box $B_{2,3}$. In sudoku 93, we then use the open triple $\{6, 8, 9\}$ at the beginning of row 4 again, but this time with respect not to row 4, but to box $B_{2,1}$, getting sudoku 94.

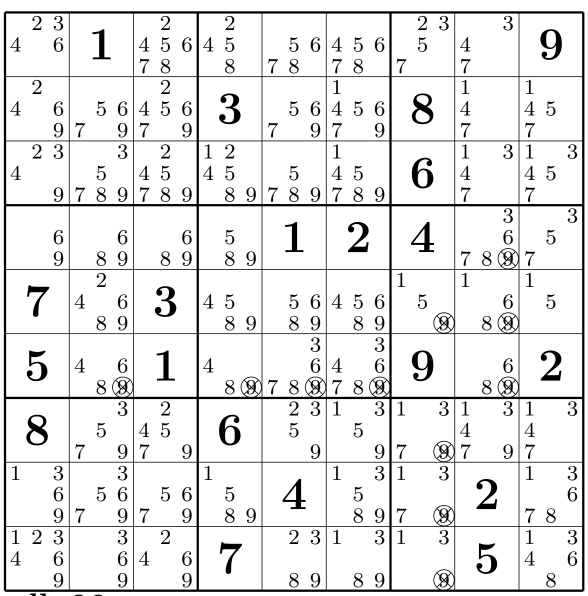


sdk 94 sdk9_17_1_XX0005

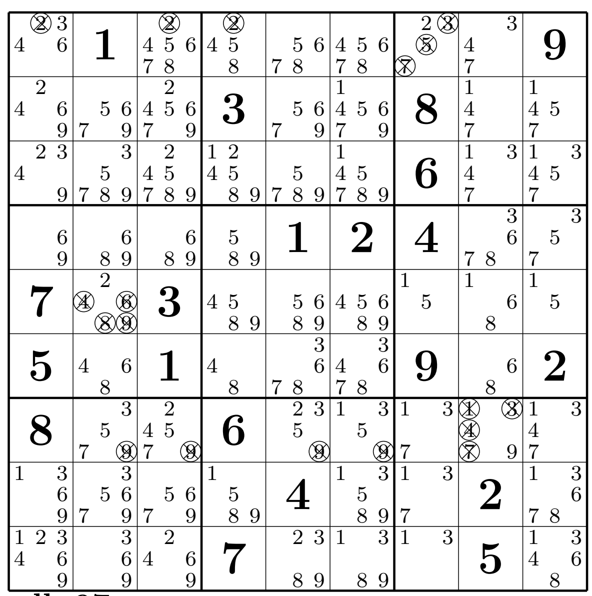


sdk 95 sdk9_17_1_XX0006

In column 2 of sudoku 94, we now see the open pair $\{4, 2\}$, which allows us to eliminate these candidates in the remaining cells of column 2. In sudoku 95, we then use the open quadruple $\{1, 3, 4, 7\}$ at the beginning of column 8, getting sudoku 96.

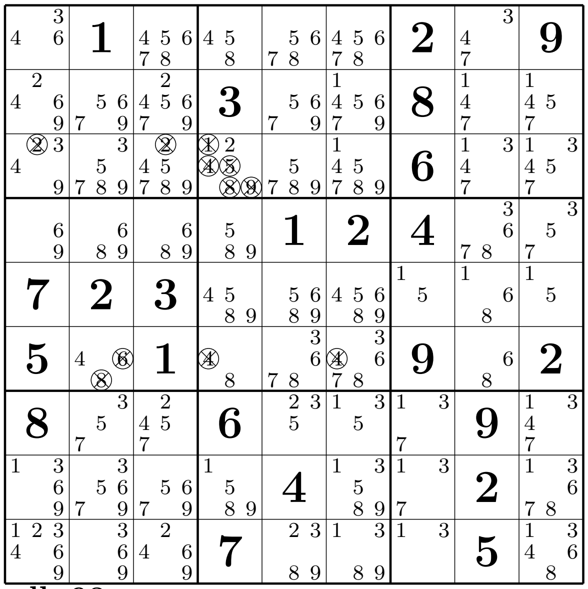


sdk 96 sdk9_17_1_XX0008

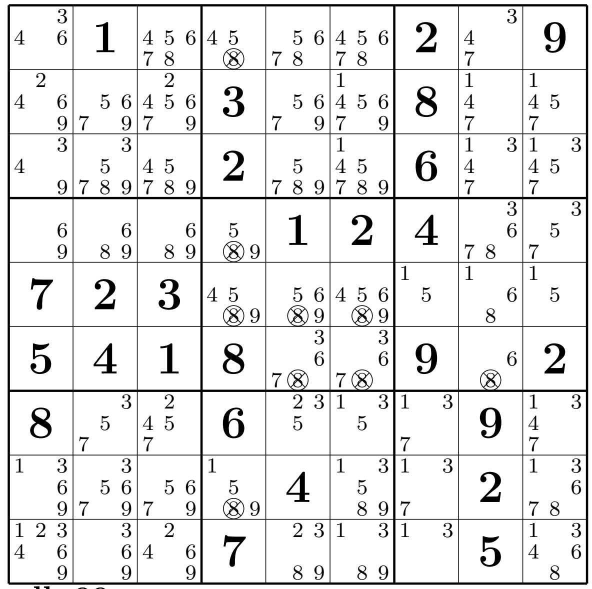


sdk 97 sdk9_17_1_XX0009

In box $B_{2,3}$ of sudoku 96, we exploit the open pair $\{6, 8\}$ to get sudoku 97. From then on to the end (sudoku 101), we exclusively apply rule F for candidate elimination. In sudoku 97, cells $(6, 2)$ and $(6, 7)$ turn out to have only one candidate left (4 and 9, respectively).

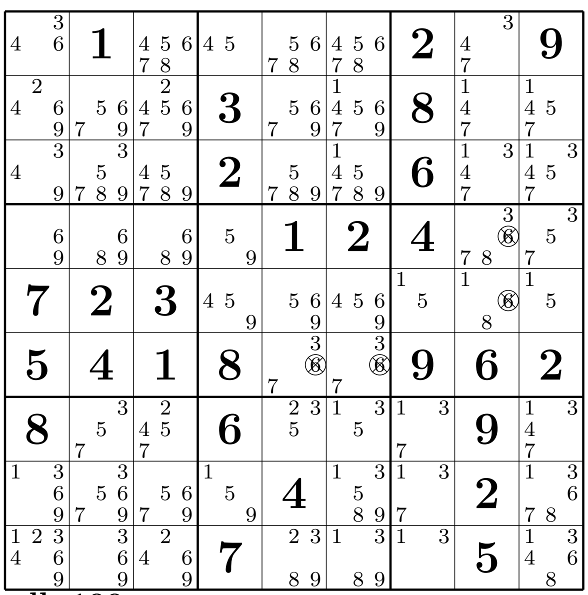


sdk 98 sdk9_17_1_XX0010

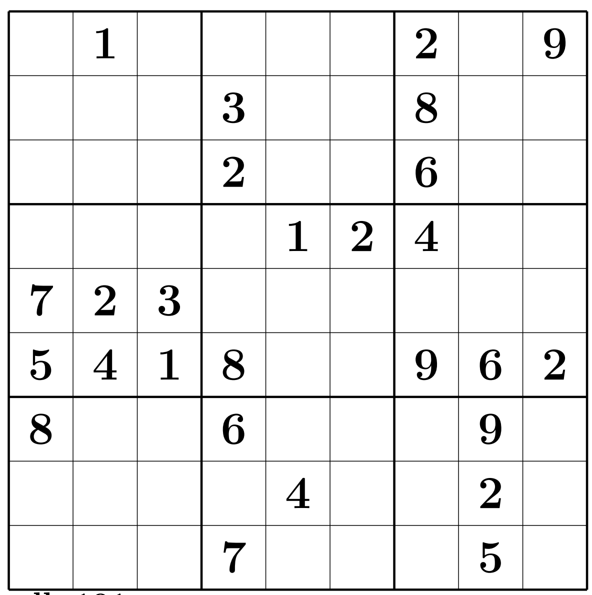


sdk 99 sdk9_17_1_XX0011

In sudokus 98 and 99, cells $(6,4)$ and $(6,8)$ have only one candidate left (8 and 6, respectively).



sdk 100 sdk9_17_1_XX0012



sdk 101 sdk9_17_1_XX0013

In sudoku 100 , finally, rule F can be applied to cell $(5,8)$, in which the only remaining candidate is 8. Now sudoku 101 is elementary, i.e. can be completed by FN alone.

Remark. The previous sudokus up to and including 100 are not elementary. This might seem to contradict the observation that from sudoku 97 on, we only used the elementary rule F. However, if we start out to complete, for instance, sudoku 100 "from scratch", we loose the previous reductions of the candidate list.

Rule 3 (Tuple reduction T)

Rule T can be split up into three subrules T_B, T_R, and T_C:

- T_B *The candidates appearing in an open tuple lying within a common box can be eliminated in the remaining unoccupied cells of the box.*
- T_R *The candidates appearing in an open tuple lying within a common row can be eliminated in the remaining unoccupied cells of the row.*
- T_C *The candidates appearing in an open tuple lying within a common column can be eliminated in the remaining unoccupied cells of the column.*

5.2 Hidden tuples

Although the process of tuple reduction is defined solely by reference to *open* tuples, it can sometimes be made more convenient by the use of *hidden* tuples.

Definition 7 (Hidden tuples) *>>hidden<<*

Let t be a tuple consisting of *all* the unoccupied cells with respect to some box, or row, or column. Suppose that t contains a subtuple t_1 of k cells such that some k candidates occurring in t are restricted to t_1. Then t_1 is called a *hidden tuple*, and the remaining $n-k$ cells of t neccessarily form an open tuple containing only the remaining $n-k$ candidates. Therefore, these remaining candidates can be eliminated within t_1. Thus t splits up into two open tuples.

Whenever a hidden tuple occurs within a set of associated unoccupied cells, the remaining unoccupied cells form an open tuple. Hidden tuples are logically redundant. But it is often easier to detect a hidden pair than, e.g., the open triple, quadruple, quintuple accompaning it. In sudoku 92 of the above example, in box $B_{2,2}$ we have a hidden tuple $\{3,7\}$ which would lead to the elimination of candidates 4, 6, 8, 9 in cells $(6,5)$ and $(6,6)$. The same effect would be produced by exploiting the open quadrupel $\{4,6,8,9\}$ in the remaining unoccupied cells of box $B_{2,2}$.

Whenever some unoccupied cells c_1, ..., c_k are maximal with respect to a box (or a row, or a column), i.e., there are no more unoccupied cells in that box (or row, or column), then iterating rule T has the effect of splitting these cells up into subsets of irreducible open tupels. As a matter of fact, the elementary rules are special cases of tuple reduction, either of *open* (F), or of *hidden* (N_B, N_R, N_C) *singletons.* If the reduction process ends up with only open singletons, then the sudoku is completed.

Example 5.2 (Open and hidden tuples) *>>ohtuples<<*

This example illustrates the interplay between open and hidden tuples. By FN, we get the sudoku below right from the sudoku to the left:

	5	1			6			
		4						9
							7	4
2			8		3			
			2		9			3
9	2							
6						5		
			7			4	8	

sdk 102 sdk9_NZZaS_161011_trsf

7	5	1	9	4	6	2 3	2 3	8
8	6	4	3	2	7	1	5	9
3	9	2	5	8	1	6	7	4
2	1 7	5 6 9	8	1 5 6 7	3	7 9	4	1 5 7
1 5	1 3 7 8	3 5 6 9	4 6	1 5 6 7	4 5	2 7 8 9	1 2 6	1 5 7
4	1 7 8	5 6	2	1 5 6 7	9	7 8	1 6	3
9	2	8	4 6	5 6	4 5	3 7	1 3	1 7
6	4	7	1	3	8	5	9	2
1 5	1 3	3 5	7	9	2	4	8	6

sdk 103 sdk9_NZZaS_161011_trsf_FN

In row 5 of sudoku 103, we find the hidden tuple $\{3,8,9\}$ in cells $(5,2)$, $(5,3)$, and $(5,7)$. Therefore, the remaining candidates necessarily form an open tuple in the remaining empty cells, i.e., there is the open tuple $\{1,2,4,5,6,7\}$. This, however, is not irreducible. It can be split up into the open quintuple $\{1,4,5,6,7\}$ in cells $(5,1)$, $(5,4)$, $(5,5)$, $(5,6)$, $(5,9)$, and the open singleton $\{2\}$ (also found by N_R) in cell $(5,8)$. Conversely, $\{2,3,8,9\}$ is a hidden quadrupel in cells $(5,2)$, $(5,3)$, $(5,7)$, and $(5,8)$ (which is, of course, not irreducible). For completeness, we add that the whole row can be viewed as an open 9-tuple.

Quite independently from the order in which we exploit open and hidden tuples, row 5 is finally split up into the irreducible open tuples $\{2\}$, $\{3,8,9\}$, and $\{1,4,5,6,7\}$. Then completion can be attained by rule F alone.

5.3 Specifying the use of tuples

We may give more detailed information on the application of the T rules by writing, for example, T_4^2. By this, we mean that we take into consideration open tuples of size up to 4 and hidden tuples of size up to 2. If we are just exploiting open pairs, we may write T_2^0 or T_2. Application of hidden tuples up to size 3 can be indicated by T_0^3 or T^3.

5.4 Problems

The six sudokus below can be completed by FN and open pairs (FN+T_2):

	4						1	
9			4		7			5
			1		6			
	7	6				4	3	
				2				
	2	8				5		
			3		9			
7			5		1			6
	8						7	

sdk 104 sdk9_ta_090708S_Z68_trsf

	9							
	4	1	6	8				5
				7			4	
							7	
	1	8				3	9	
	2							
	5			2				
2	6			5	9	4	8	
							3	

sdk 105 sdk9_ta_200508S_Z28_trsf

		5				2		
	6		9		4		8	
8								1
	4				8		1	
				6				
	1		4		2		3	
5								3
	9		3		1		7	
		2				9		

sdk 106 sdk9_ta_210508S_Z44_trsf

9		4						2
		5		6		3		
				2				
		7			4	5	6	
			5		6			
	3	6	8			1		
				5				
		1		7		6		
8						2		1

sdk 107 sdk9_ta_300708S_Z55_trsf

1			2		3			5
		5		9	8	3		
	5	2				7	8	
	6						5	
	9	3				2	1	
		8	9	7	4	5		
6			8		2			9

sdk 108 sdk9_20min_020909M_Z14_trsf

		2				9		
			9		5			
	3		2		6		8	
	8					6	3	
				6				
	6	5				8	9	
	5		6		9		2	
			3		8			
		7				4		

sdk 109 sdk9_tbz_210710_Z41_trsf

The following four sudokus can be completed by FN and hidden pairs (FN+T^2). Alternatively, they can be completed by FN and open pairs and triples (FN+T_3):

		3		8		4		
9								7
8			7					2
			4		7	6		
5								3
		6	1		3			
7					9			
4								9
		1		6		8		

sdk 110 sdk9_ta_091009_Z97_trsf

	2			4			5	
3						7		8
	7						3	
			3		5			
5								1
			8		1			
	3						7	
8		2				1		9
	9			6			8	

sdk 111 sdk9_NEWS_290508S_Z21_trsf

	2		1		4		3	
4				3				8
		6				9		
1								9
	5			2			8	
9								
		7				5		
8				5				3
	1		6		2		9	

sdk 112 sdk9_NEWS_010709_Z67_trsf

		2				7		
		6	5				4	
1	7			8				6
	5				2			
		3		6		2		
			9				7	
3				4			9	1
	9				5	6		
						3		

sdk 113 sdk9_NEWS_230709_Z71_trsf

The following two sudokus can be completed by FN and hidden pairs (FN+T^2). Alternatively, they can be completed by FN and open pairs, triples, and quadrupels (FN+T_4):

		3		5		4		
	8		1		4		9	
7								2
	5						6	
3								9
	2						1	
6								8
	1		6				5	
		7		4		6		

sdk 114 sdk9_ta_170510_Z86_trsf

	7				6			
4			1			8		
					2		9	
	1				3	7		4
				8				
6		5	7				3	
	3		5					
		9						2
			4				6	

sdk 115 sdk9_NEWS_150508_Z86_trsf

The following two sudokus can be completed by FN, open and hidden pairs (FN+T_2^2). Alternatively, they can be completed by FN and open triples (FN+T_3).

	4		5				1	
7			2					4
		2		9				
9	3							
				6		3		
							2	8
				1		9		
4					9			7
	7				3		6	

sdk 116 sdk9_BaA_250711_Z51_trsf

	1	8		2				
					9			4
		4		3		6		5
	3							
7				5		9		3
							1	
2		3		4		8		
5			1					
				7		5	4	

sdk 117 sdk9_tbz_270411_Z51_trsf

The following two sudokus can be completed by FN and hidden triples (FN+T^3). Alternatively, they can be completed by open tupels of length up to 4 (FN+T_4), and up to 5 (FN+T_5), respectively.

		1	6		7	8		
6			3	5				9
5						4		3
		2		4		1		
8		9						6
4				6	3			8
		7	9		2			

sdk 118 sdk9_ta_301109_Z79_trsf

	5	1			6			
		4						9
							7	4
2			8		3			
			2		9			3
9	2							
6						5		
			7			4	8	

sdk 119 sdk9_NZZaS_161011_trsf

The last problem is the sudoku of example 5.2.

The following four sudokus require, aside from FN, rule B as well as rule T.

7			9		1			5
	3			6			2	
9				3				6
	7		4		6		3	
8				7				9
	2			4			1	
1			5		9			

sdk 120 sdk9_tbz_070911_Z77_trsf

7			2		5		8	4
1		6		9			7	
	7		9				4	
				8				
	4				1		3	
	9			6		5		1
5	1		8					3

sdk 121 sdk9_tbz_170811_Z86_trsf

		8			6		2	
		6	9					4
1	4	9		7				
	2							1
		5				3		
4							8	
				6			3	5
3					5	4		
	9		1			8		

sdk 122 sdk9_BaA_250809_Z99_trsf

		1			7	8		
6			3	5				9
5						4		3
		2		4		1		
8		9						6
4				6	3			8
		7	9		2	5		

sdk 123 sdk9_ta_301109_Z34_trsf

6 X-Chains (One-Candidate Chains)

6.1 Cell chains

Definition 8 (Cell Chain) *>>cellchain<<*

By a *cell chain* we understand a sequence of cells such that every two consecutive cells are associated.

A cell chain is *cyclic* if some cell appears more than once in it, *non-cyclic* else.

Although cell chains can be investigated as a topic in itself, we always presuppose the presence of a candidate table. Therefore, every cell chain automatically corresponds to a unique *sequence of candidate sets*.

6.2 Strong and weak edges

Any two associated cells determine an *edge*. If we feel more comfortable with a precise definition, an *edge* is just a pair of associated cells. Although we might think of an ordered pair (a directed edge), this is not necessary for our purpose. So we can think of an edge as of an *unordered pair*, or 2-element set, of associated cells.

Definition 9 (Weak and strong edges) *>>weakstrong<<*

An edge is said to be

(i) *weak* with respect to a candidate, if there is a third cell containing the candidate which is associated with both end cells of the edge,

(ii) *strong* otherwise.

6.3 X_1-Chains

We define x_1*-chains* inductively, having in mind some particular candidate c:

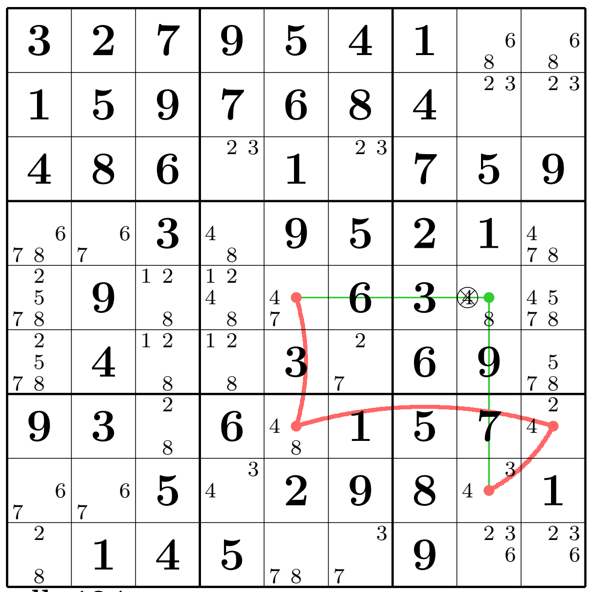
Definition 10 *Any chain consisting of an odd number of strong edges is an* x_1*-chain. If two* x_1*-chains are connected by a weak edge, then the resulting chain is also an* x_1*-chain.*

Then it follows by straightforward induction that in any completion, the candidate will be assigned to at least one of the end cells of an x_1-chain. Therefore, we have the following rule:

Rule 4 (X_1) *>>x1<<* *If, with respect to some candidate, a sequence of cells form an* x_1*-chain, then the candidate can be eliminated from any cell associated with both end cells of the chain.*

Remark: X_1-Cycles. In FOWLER[2], the end cells associating the additional cell with both ends of the chain are included to form a cycle. Distinction is made between x_1-cycles and x_3-cycles. In x_1-cycles, the links are of the same kind, e.g. both with respect to a box, or a row, or a column. In x_3-cycles, they are of different kinds, e.g. one with respect to a row, the second with respect to a column.

In the example below, we find an x_1-chain with respect to candidate 4. (There is even a second x_1-chain, which we will, however, leave aside.)

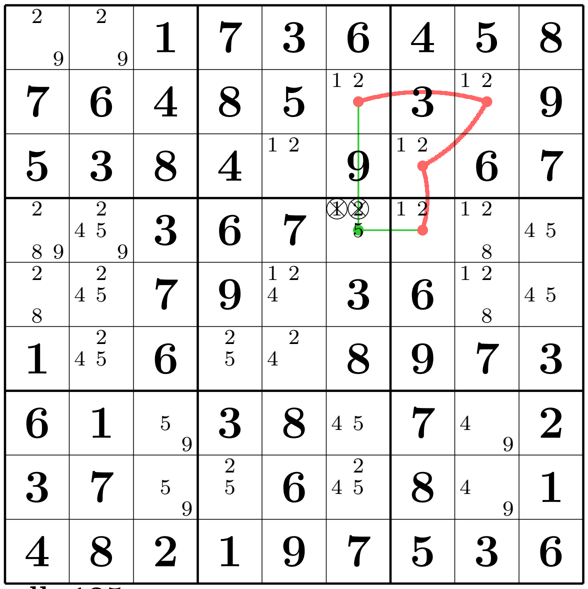


sdk 124 sdk9_spp136_plus

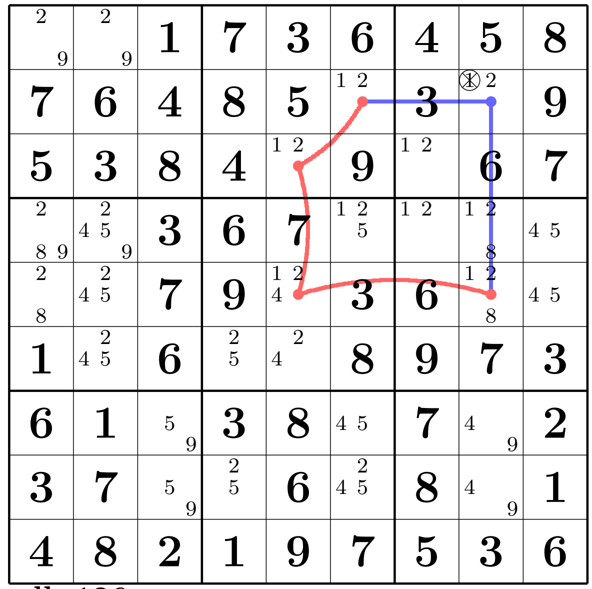
Therefore, candidate 4 can be eliminated from cell (5,8). Completion then only requires FN.

Example 6.1 (Remote pairs)

The technique of *remote pairs* can be replaced by twice applying rule 4 (X_1). In order to explain this, we take the example from SADMAN[8]:



sdk 125 sdk9_sad_rp_1



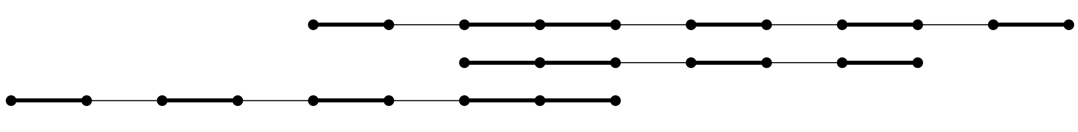
sdk 126 sdk9_sad_rp_1

The chain on the left connects 4 cells with candidate set $\{1, 2\}$. The number of edges being odd, we have so-called remote pairs, which allows us to eliminate both candidates from cell $(4, 6)$.
However, there is no need for a special rule. The chain in question is simply an x_1-chain with respect to candidate 1 as well as to candidate 2. Therefore, both can be eliminated from cell $(4, 6)$ by rule 4 (X_1).
The chain on the right is, however, only an x_1-chain with respect to candidate 1, because only one of the three edges is strong with respect to 2.
Now 5 has to be assigned to cell (4,6), and 2 to cell (2,8). Completion then only requires FN.

6.4 X_2-Chains

Definition 11 *An* x_2-chain *with respect to some candidate is a chain consisting of a subchain of an* even *number of strong edges and an arbitrary number of subchains each consisting of an* odd *number of strong edges with respect to the given candidate, whereby any two of these subchains are connected by exactly one weak edge.*

The following figur shows schematically some possibilities for x_2-chains.



Thick lines mean strong, thin lines mean weak edges. The subchain of an even number of strong edges is represented by two strong edges. Every isolated strong edge can be replaced by a subchain of an odd number of strong edges, but every weak edge remains exactly one weak edge.

Rule 5 (X_2) >>*x2*<< *If the end cells of an x_2-chain are associated, then in both end cells of the subchain formed by the even number of strong edges the candidate can be eliminated.*

For assume that the candidate be assigned to one of the end cells of the subchain with the even number of strong edges. Then it would have to be assigned to the other end cell of this subchain as well as to both end cells of the complete chain. But this is a contradiction, as these end cells are supposed to be associated.

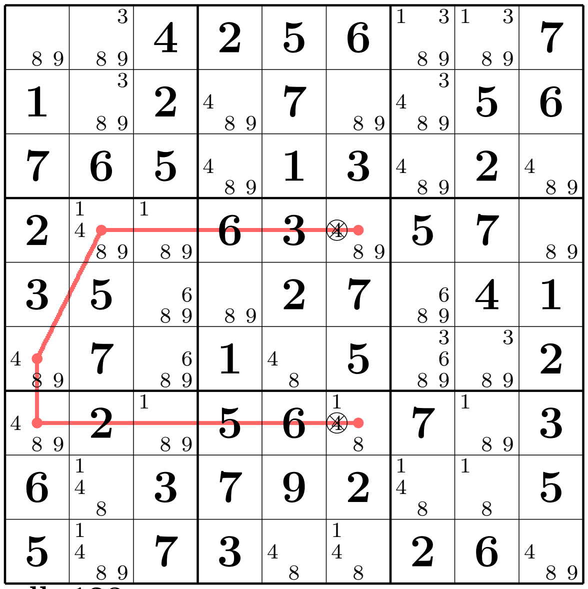
Remark: X_2-Cycles. If the end cells of an x_2-chain are associated, then the connecting edge may be weak or strong. If it is included, the chain is completed into a cycle, which in FOWLER[2] is called an x_2-cycle. Any x_2-cycle breaks down into an x_1-chain, a chain of an even number of strong edges, and two weak edges connecting the two. Then rule X_2 can be easily inferred from rule X_1 and the fact that in a chain of an even number of strong edges, the candidate will have to be set either at both edges, or at none of them.

The following example is taken from SADMAN[9]. Candidate 4 in cell (2,6) can be removed by rule $C \succ B$. (In column 4, candidate 4 is restricted to box $B_{1,2}$.)

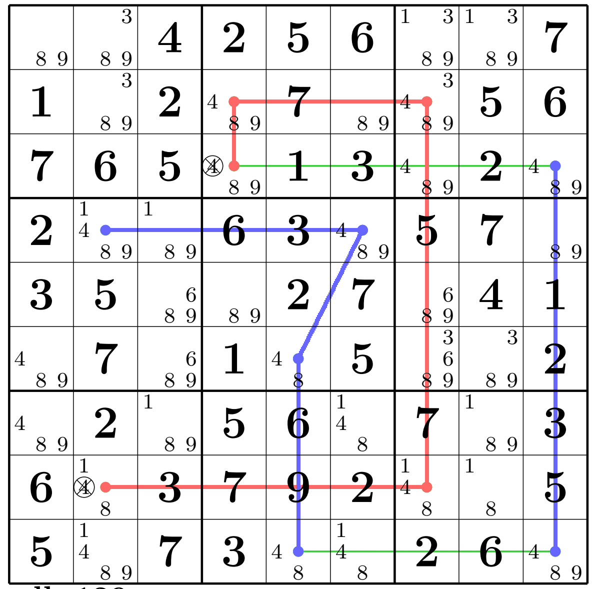
8 9	3 8 9	**4**	**2**	**5**	**6**	1 3 8 9	1 3 8 9	**7**
1	3 8 9	**2**	4 8 9	**7**	⊗4 8 9	3 4 8 9	**5**	**6**
7	**6**	**5**	4 8 9	**1**	**3**	4 8 9	**2**	4 8 9
2	1 4 8 9	1 8 9	**6**	**3**	4 8 9	**5**	**7**	8 9
3	**5**	6 8 9	8 9	**2**	**7**	6 8 9	**4**	**1**
4 8 9	**7**	6 8 9	**1**	4 8	**5**	3 6 8 9	3 8 9	**2**
4 8 9	**2**	1 8 9	**5**	**6**	1 4 8	**7**	1 8 9	**3**
6	1 4 8	**3**	**7**	**9**	**2**	1 4 8	1 8	**5**
5	1 4 8 9	**7**	**3**	4 8	1 4 8	**2**	**6**	4 8 9

sdk 127 sdk9_sad_col_3

Then for candidate 4 exist no less than 5 x_2-chains. Here are two of them:



sdk 128 sdk9_sad_col_3a



sdk 129 sdk9_sad_col_3a

The x_2-chain in the sudoku on the left simply consists of 4 strong edges, and the end cells are column-associated. Therefore by rule X_2, candidate 4 can be eliminated from cells $(4,6)$ and $(7,6)$. Now rule N_C says that we have to set cell $(9,6)$ to 4. Completion then only requires FN.

The fancy chain on the right would allows us to eliminate candidate 4 from cells (3,4) and (8,2). Then by rule N_C, we could assign digit 4 to cell(2,4), and by rule N_R, digit 4 to cell (8,7). This alone would not yet, however, leave a sudoku which could be completed by FN alone. We would still have to make use of the chain in the sudoku on the left.

6.5 Problems

The two sudokus below can be completed by using methods FNBT and an x_1-chain:

	5				3			7
6			8	5				
						1		
	7							2
8	9			4			6	3
3							7	
		8						
				3	2			9
9			7	6			5	

sdk 130 sdk9_tbz_310310_Z35_trsf

			6			5		
	2		3			4	1	
5	6			7				
							3	9
		5				6		
8	9							
				3			9	7
		1			4		6	
		8			5			

sdk 131 sdk9_tbz_290910_Z82_trsf

The next two sudokus can be completed by using methods FNBT and an x_2-chain:

		8				1		
3	9						2	8
	7		8		9		4	
4				7	2			6
2			5	1	8			9
	2		6		3		5	
9	1						6	4
		7				9		

sdk 132 sdk9_Knaur_128_Z44_trsf

			1			2		
	8				6		1	
5					9			
	5	9	8		4			3
				9				
7					5	4	2	
			3					4
	7		4				6	
		2			8			

sdk 133 sdk9_tbz_120510_Z64_trsf

The next two sudokus can be completed by using methods FNBT and an x_1- as well as an x_2-chain (not necessarily in this order):

	4						3	
5				1				4
		1	7			8		
			9		7	1		
	7			6			2	
		6	2		8			
					6	7		
4				5				8
	9						4	

sdk 134 sdk9_tbz_221210_Z91_trsf

			3			4		
					2			
	9			1		6		5
7							2	
		3		4		7		
	5							6
2		5		9			8	
			6			1		
		7			8			

sdk 135 sdk9_tbz_241110_Z21_trsf

6.6 Hints to the problems

By FNB and $FNBT_2$, respectively, from the first two sudokus we get:

1 2 4	**5**	1 2 4 9	1 2 4 9	1 2 9	**3**	**6**	**8**	**7**
6	1 2 3	**7**	**8**	**5**	1 9	2 3	2 3 9	**4**
2 4	**8**	2 3 4 9	2 4 6 9	**7**	4 6 9	**1**	2 3 9	**5**
4 5	**7**	4 5 6	**3**	**8**	5 6 9	4 9	**1**	**2**
8	**9**	1 2	1 2	**4**	**7**	**5**	**6**	**3**
3	1 2 4 6	1 2 4 5 6	1 2 5 6 9	1 2 9	1 5 6 9	4 9	**7**	**8**
1 2 5	1 2 3	**8**	1 4 5 9	1 9	1 4 5 9	**7**	2 3	**6**
7	1 6	1 5 6	1 5	**3**	**2**	**8**	**4**	**9**
9	2 3 4	2 3 4	**7**	**6**	**8**	2 3	**5**	**1**

sdk 130 sdk9_tbz_310310_Z35_trsf_e

Sudoku 130 contains the chain ((2,7), (9,7), (7,8), (7,2)), which is an x_1-chain with respect to candidate 3. Therefore by rule 4, candidate 3 can be eliminated from cell (2,2). As an immediate consequence, cell (3,3) has to be put to 3. Then completion is possible by FNB.

1	**8**	3 4	**6**	2 4	**9**	**5**	**7**	2 3
7	**2**	**9**	**3**	**5**	**8**	**4**	**1**	**6**
5	**6**	3 4	1 2 4	**7**	1 2	**9**	2 8	2 3 8
2 6	**4**	2 6 7	**5**	**8**	1 2 6	1 2 7	**3**	**9**
3	**1**	**5**	2 4 9	2 4 9	**7**	**6**	2 4 8	2 4 8
8	**9**	2 6 7	1 2 4	1 2 4 6	**3**	1 2 7	**5**	1 2 4
4	**5**	2 6	1 2 8	**3**	1 2 6	1 2 8	**9**	**7**
2 9	3 7	**1**	7 8	2 9	**4**	3 8	**6**	**5**
2 6 9	3 7	**8**	1 2 7 9	1 2 6 9	**5**	1 2 3	2 4	1 2 4

sdk 131 sdk9_tbz_290910_Z82_trsf_e

Sudoku 131 contains the chain ((4,1), (9,1), (9,5), (7,6)), which is an x_1-chain with respect to candidate 6. Therefore, candidate 6 can be eliminated from cell (4,6). As an immediate consequence, cells (6,5) and (7,6) have to be set to 6 (rules N_B and N_C). Then completion is possible by FN alone.

The next two sudokus can be extended by $FNBT_2^2$ and $FNBT_4$, respectively, to

5 6	**4**	**8**	2 3	2 3	5 6	**1**	**9**	**7**
3	**9**	5 6	1 7	**4**	1 7	5 6	**2**	**8**
1	**7**	**2**	**8**	5 6	**9**	3 5 6	**4**	3 5
4	5 8	1 9	3 9	**7**	**2**	3 5 8	1 3 8	**6**
7	5 8	1 9	3 4 9	3 6	4 6	2 3 5 8	1 3 8	2 3 5
2	3 6	3 6	**5**	**1**	**8**	**4**	**7**	**9**
8	**2**	**4**	**6**	**9**	**3**	**7**	**5**	**1**
9	**1**	3 5	2 7	2 5 8	5 7	2 3 8	**6**	**4**
5 6	3 6	**7**	1 4	2 5 8	1 4	**9**	3 8	2 3

sdk 132 sdk9_Knaur_128_Z44_trsf_e

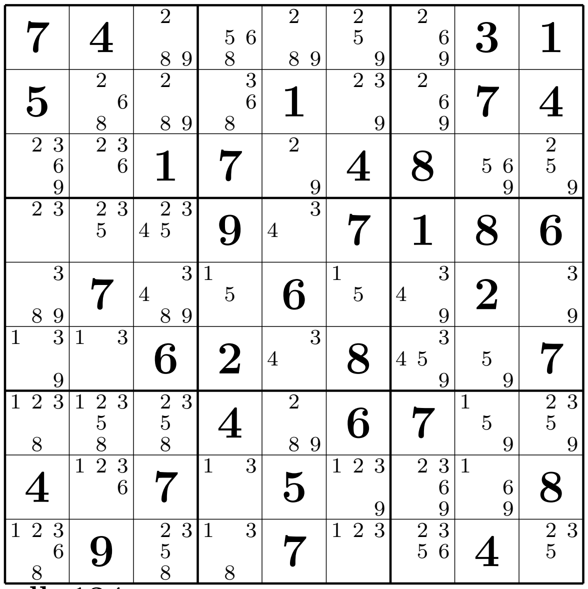
3 6	**9**	6 7	**1**	4 8	3 7	**2**	4 5 8	5 8
2 3	**8**	**4**	**5**	2 3	**6**	**9**	**1**	**7**
5	1 2	1 7	2 7	4 8	**9**	3 6	3 4 8	6 8
1 2 6	**5**	**9**	**8**	1 2	**4**	1 6	**7**	**3**
2 4	2 3 4	1 6	2 7	**9**	3 7	1 5 6	5 8	1 5 6 8
7	1 3	**8**	**6**	1 3	**5**	**4**	**2**	**9**
8	1 6	**5**	**3**	6 7	**2**	1 7	**9**	**4**
9	**7**	**3**	**4**	**5**	**1**	**8**	**6**	**2**
1 4	1 4 6	**2**	**9**	6 7	**8**	1 3 5 7	3 5	1 5

sdk 133 sdk9_tbz_120510_Z64_trsf_e

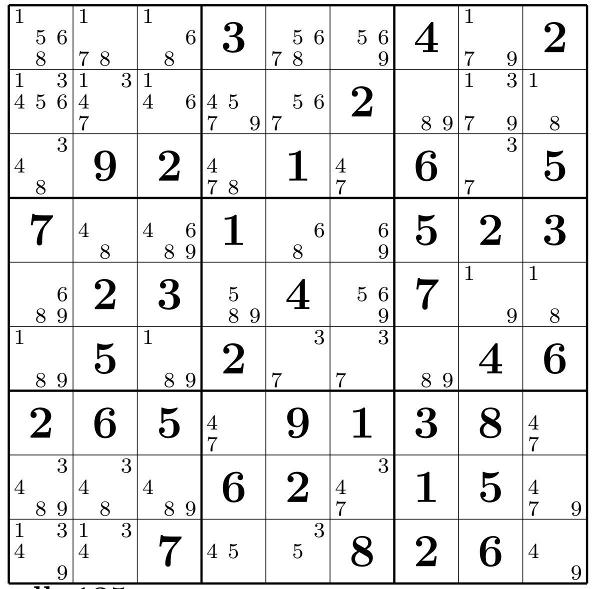
Sudoku 132 contains the chain $((3,5), (1,6), (1,1), (9,1), (9,5))$, which is an x_2-chain with respect to candidate 5. It is a "simple" x_2-chain in the sence that it contains no weak edges, but just consists of 4 strong edges with respect to 5. Therefore by rule 5, candidate 5 can be eliminated from cells $(3,5)$ and $(9,5)$. As an immediate consequence, cell $(3,5)$ can be put to 6. Then completion is possible by F alone.

Sudoku 133 contains the chain $((7,7), (7,2), (3,2), (3,3), (5,3), (5,9), (9,9))$, which is an x_2-chain with respect to candidate 1. This chain is not "simple". The core strong-edge part is $((3,2), (3,3), (5,3))$. At both ends, a weak edge is added, and then one single additional strong edge. Therefore, candidate 1 can be eliminated from cells $(3,2)$ and $(5,3)$, which are the ends of the core strong-edge part. As an immediate consequence, cells $(3,2)$ and $(5,3)$ can be set to 2 and 6, respectively. Then completion is possible by F alone.

The next two sudokus can be extended by $FNBT_2$ and $FNBT_3$, respectively, to



sdk 134 sdk9_tbz_221210_Z91_trsf_e



sdk 135 sdk9_tbz_241110_Z21_trsf_e

Sudoku 134 contains the chain ((3,1), (9,1), (8,2), (8,8), (3,8)), which is an x_2-chain with respect to candidate 6. The core even-edged part is ((3,1), (9,1), (8,2)), and therefore, candidate 6 can be eliminated from cells (3,1) and (8,2). Therefore, cell (9,1) can be set to 6 by rule N_B. As now candidate 8 disappears from this cell, the edge ((9,3), (9,4)), which was a weak edge for candidate 8, now becomes a strong edge, and therefore ((1,5), (7,5), (9,4), (9,3)) turns into an x_1-chain for candidate 8. This leads to the elimination of candidate 8 from cell (1,3). Completion can then be achieved by FNB.

Sudoku 135 contains the chain ((1,8), (1,6), (2,4), (5,4)), which is an x_1-chain with respect to candidate 9. Therefore, candidate 9 can be eliminated from cell (5,8), which implies that this cell can be set to 1, and cell (5,9) can be set to 8. Now by rule N_B, cells (4,5) and (3,4) can be both set to 8. Edge ((2,4), (3,6)) becomes a strong edge with respect to candidate 4. By rule F, cell (4,2) can be set to 4, which makes ((2,3), (8,3)) a strong edge with respect to candidate 4. The chain ((2,4), (3,6), (8,6), (8,3), (2,3)) turns into an x_2-chain with the core strong-edged part ((2,4), (3,6), (8,6)). By rule X_2, candidate 4 can be eliminated from cells (2,4) and (8,6). Then completion only requires rule F.

7 Pair Chains (Y-Chains)

`>>ychains<<` In the presence of a candidate table, to every cell chain corresponds a sequence of candidate sets. If this sequence is made up of pairs, we call the cell chain a *pair chain*. Because in a chain, any two consecutive cells are associated, every possible assignment to a chain is a sequence of candidates such that any two consecutive members are distinct. This leads us to define:

Definition 12 (Homogeneous, strictly inhomogeneous sequences) *`>>defhominh<<`*

(i) We call a sequence *homogeneous* if all elements are equal.

(ii) We call a sequence *strictly inhomogeneous* if no two consecutive elements are equal.

By this definition, the terms *homogeneous* and *strictly inhomogeneous* can be applied to candidate sequences as well as to sequences of (candidate) sets. Note that $(\{1,2\}, \{1,3\}, \{1,2\})$ is strictly inhomogeneous, although the first and the last pair are equal. We now turn our attention to the case where all candidate sets are (unordered) pairs, i.e. consist of exactly two distinct candidates.

7.1 Y-sequences and y-chains

Definition 13 (Y-sequence, y-chain) *`>>yseqchain<<`*

A *y-sequence with respect to* c_0 is a sequence of pairs $(\pi_1, \ldots, \pi_n)$ such that, for some strictly inhomogeneous sequence $(c_0, c_1, \ldots, c_{n-1}, c_0)$, $\pi_i = \{c_{i-1}, c_i\}$ for $i = 1, \ldots, n-1$, and $\pi_n = \{c_{n-1}, c_0\}$, i.e. a pair sequence that can be written in the form

$$\Pi = (\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-2}, c_{n-1}\}, \{c_{n-1}, c_0\}).$$

A *y-chain with respect to some candidate* c_0 is a pair chain such that the corresponding sequence of candidate sets is a y-sequence with respect to c_0.

The wording "can be written in the form" reminds of the fact that candidate sets are *unordered* pairs (2-element sets). Therefore, $\{a, b\} = \{b, a\}$ for any a, b. For *ordered* pairs, however, $(a, b) \neq (b, a)$ if $a \neq b$. Note that braces are used for (unordered) pairs, and parentheses for ordered pairs. If candidate c_0 is *not* assigned to the first cell, then the assignments necessarily are

$$c_1, c_2, \ldots, c_{n-1}, c_0,$$

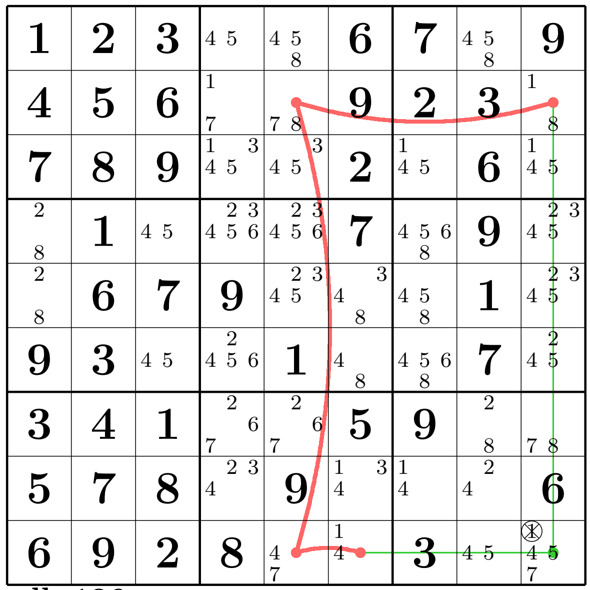
i.e. c_0 is assigned to the last cell. As an immediate consequence, we obtain:

Rule 6 (Y) *`>>ruley<<`* *If, with respect to some candidate, a sequence of cells forms a y-chain, then in any cell associated with both ends of the chain, the candidate can be eliminated.*

Sometimes, e.g. in GoldenChain[7], y-chains are named *golden chains*. The simplest y-chain is usually called *xy-wing*. It consists of exactly two edges.

Example 7.1

A y-chain occurs, for example, in "puzzle y - 1" of FOWLER[2]:



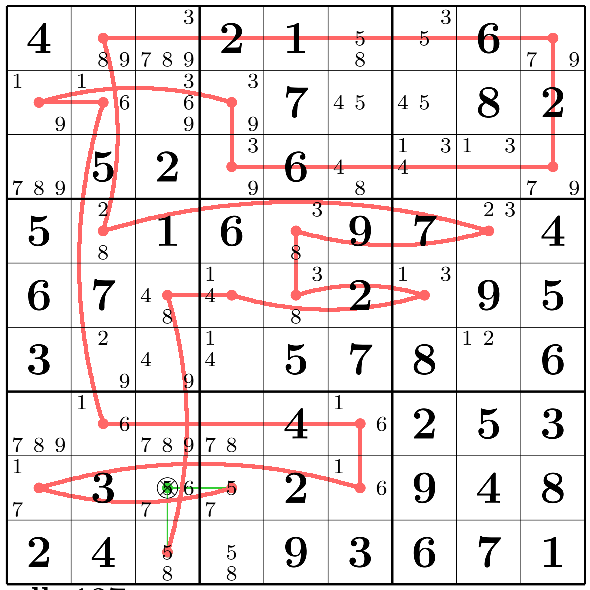
sdk 136 sdk9_att_3

The marked cell chain is $((2,9),\ (2,5),\ (9,5),\ (9,6))$. It is a y-chain with respect to candidate 1, as the corresponding pair sequence can be written as $(\{1,8\},\{8,7\},\{7,4\},\{4,1\})$. Hence by rule 6, candidate 1 can be eliminated from cell (9,9), which is associated to both ends of the chain. Then by rules N_B and N_R, cells $(8,7)$ and $(9,6)$ can both be set to 1. The resulting sudoku is, however, not yet elementary.

Yet there is also the y-chain $((2,4),\ (2,9),\ (7,9))$, to which corresponds the y-sequence $(\{7,1\},\ \{1,8\},\ \{8,7\})$. It allows us to eliminate candidate 7 from cell $(7,4)$. Then by rule N_C, cell $(2,4)$ has to be set to 7. Now the resulting sudoku is elementary.

Example 7.2 *>>fowler4<<*

A rather intricate example of a y-chain is presented in FOWLER[2] ("puzzle y - 2"):



sdk 137 sdk9_att_4

It starts at cell (8,4) and ends at cell (9,3). It comprises 19 edges, and therefore 20 cells. The corresponding sequence of candidate pairs can be written as

$$(\{5,7\}, \{7,1\}, \{1,6\}, \{6,1\}, \{1,6\}, \{6,1\}, \{1,9\}, \{9,3\}, \{3,9\}, \{9,7\}, \{7,9\}, \{9,8\}, \{8,2\}, \{2,3\}, \{3,8\}, \{8,3\}, \{3,1\}, \{1,4\}, \{4,8\}, \{8,5\}). \quad (1)$$

`>>intric<<`

It is a y-sequence with respect to candidate 5. Hence by rule (6), this candidate can be eliminated from cell $(8,3)$.

As a matter of fact, the above sudoku can be completed without recourse to this chain, which is very hard to spot. Indeed, the cell chain $((1,6),(1,7),(5,7),(5,4),(5,3))$ turns out to be a y-chain for candidate 8, which therefore can be eliminated from cell $(1,3)$. The corresponding pair sequence can be written as $(\{8,5\}, \{5,3\}, \{3,1\}, \{1,4\}, \{4,8\})$. Likewise, $((3,9),(3,4),(2,4),(2,1),(8,1))$ is a y-chain for candidate 7, which can be eliminated from cell $(3,1)$. Then the solution can be completed by elementary methods alone.

If we omit one of the four pairs $\{1,6\}$ in the pair sequence (1), then the result is no longer a y-sequence. This might suggest that it is not always clear at first sight whether or not a given pair sequence is a y-sequence.

In section 10, we give a recipe for dealing with long and intricate y-sequences.

7.2 Problems

The following 10 sudokus can all be completed by $FNBT$ and rule Y. In the first two, one single application of rule Y suffices. In the last two, iteration of $FNBT$ and rule Y is required. Especially the last sudoku contains a plethora of y-chains.

The following two sudokus can be completed by $FNBT$ and a unique application of Y:

7			4	6	5			
	1							
	3			2			9	
	9	6			1		2	
3								1
	2		8			3	5	
	5			1			3	
							6	
			2	3	9		8	7

sdk 138 sdk9_tbz_270706_Z32_trsf

6	7							1
		1				4		8
	8				2		9	
		7		1				
			4	6	7			
				5		2		
	3		8				6	
4		6				5		
7								9

sdk 139 sdk9_tbz_101110_Z78_trsf

In the next two sudokus, FNB leads to sudokus each with 3 y-chains. In each case, application of rules Y and F lead to completion.

		2				9		
			9		5			
	3		2		6		8	
	8	4					3	
				6				
	6	5				8	9	
	5		6		9		2	
			3		8			
		7				4		

sdk 140 sdk9_tbz_210710_Z49_trsf

	1		8		9	5		
5				4				9
							8	
		3	7	6		1		
	9						5	
		7		9	1	2		
	7						6	
6				8				2
		4	6		7		3	

sdk 141 sdk9_Knaur_131_Z12_trsf

In the next two sudokus, FNB leads to sudokus with 2 and 3 y-chains, respectively. Rule Y has to be applied to all of them. Completion can then be reached by FN.

			8		3		7	
					1			9
				9				6
4							1	5
		1		3		2		
6	7							3
5				6				
2			9					
	1	3	7		2			

sdk 142 sdk9_tbz_280410_Z39_trsf

	2	4			6		7	
						8		
			1	9				6
4				7	8	3	5	
	5	1	6	3				4
2				4	1	7		
		3						
	4		5			9	6	

sdk 143 sdk9_Knaur_126_Z11_trsf

In the sudoku below left, FNT_3 leads to 4 y-chaines. Application of rule Y to one of them suffices to produce a sudoku which can be completed by FN. (Three of the four y-chaines have this agreeable effect.) The next sudoku requires, besides $FNBT_3$, an iterated application of rule Y.

		6		4				
1				9				
	9	3	1			2		
	1			6				
	7	4	3		2	6	8	
				7			4	
		8			9	1	7	
				8	6			3
				3		4		

sdk 144 sdk9_tbz_240706_Z24_trsf

		7		6		9		
	2		7		1		8	
								6
	4						6	
2				8				1
	5						2	
9								7
	7		8		4		9	
		3		9		5		

sdk 145 sdk9_tbz_271010_Z13_trsf

The following two sudokus require iterated applications of $FNBT$ and rule Y. The second is a real test for finding an almost endless succession of y-chains.

		8	5	6	4			
					2		4	9
	5						2	6
	9	7						
3								4
						2	9	
5	8					4	7	
1	3		7					
			4	1	5	3		

sdk 146 sdk9_Knaur_132_Z11_trsf

					4			
	6		8			1	7	9
	9			6				4
4				8		7	3	
	3	1		7				5
2				5	9		8	
3	5	8			7		6	
			6					

sdk 147 sdk9_tbz_200706_Z14_trsf

7.3 Hints to the problems

sdk 138 By FNB, the sudoku converts into the state $(62, 39)$. By this we mean that 62 cells are occupied and there remain 39 candidates in the empty cells. Then there appears the y-chain $((2,5), (2,9), (6,9))$ for candidate 9. After candidate 9 is eliminated by rule Y, completion can be achieved by rule F alone.

sdk 139 By $FNBT_2$, we reach state $(41, 100)$. Using y-chain $((5,9), (5,2), (6,3))$ for candidate 3, we can achieve completion by N_B and F.

sdk 140 By FNB, we reach state $(57, 52)$, and now there appear 3 y-chaines, all of length 3. Using any one of them, e. g. $((1,1), (1,9), (9,9))$ for candidate 8, we can then achieve completion by F.

sdk 141 By FNB, we reach state $(56, 62)$, and we have 3 y-chaines. Using, for instance, $((4,1), (5,3), (8,3), (8,6))$ for candidate 4, we can achieve completion by F alone.

sdk 142 By $FNBT_2$, we reach state $(51, 74)$. There are now the 2 y-chaines $((3,8), (3,3), (4,3), (5,1))$ and $((5,1), (9,1), (9,5), (8,6), (7,6))$, both for candidate 8. After applying rule Y to both of them, we can reach completion by N and F.

sdk 143 By FNB, we reach state $(56, 55)$. There are now 3 y-chaines: one of three edges for candidate 9, one of 4 edges for candidate 8, and one of 4 edges for candidate 9. We have to apply rule Y to all three of them. Then completion can be achieved by F alone.

sdk 144 By FNT_3, we reach state $(52, 67)$. There are now 4 y-chaines: two of three edges for candidate 5, one of three edges for candidate 9, and one of 4 edges for candidate 5. Three of them lead directly to a state where completion can be achieved by F. Apply, for instance, rule Y to the chain $((1,2), (6,2), (6,7), (5,9))$.

sdk 145 Here $FNBT$ and rule Y have to be iterated. First, $FNBT_3$ leads to $(37, 116)$. Then there is a 2-edge y-chain for candidate 3, and a 4-edge y-chain for candidate 2. After using both of them, we can get to state $(61, 44)$ by FN. Again we can spot two y-chaines: one for candidate 3 and one for candidate 5, both having 4 edges. After applying rule Y to both of them, we can get completion by rule F.

sdk 146 This sudoku can be completed by almost endlessly iterating $FNBT_2$ and Y.

sdk 147 Completing this sudoku without guessing requires extrem tenacity. We present a possibility:

Start		$(27, 196)$
$FNBT_4^2$		$(35, 124)$
$Y(6)$	$((4,9), (8,9), (9,9), (9,5, (2,5), (2,6), (6,6))$	$(35, 123)$
NBT_2		$(37, 110)$
$Y(1)$	$((1,5), (5,5), (8,5), (7,4), (7,7), (7,2))$	$(37, 109)$
BT_2		$(37, 107)$
$Y(8)$	$((3,7), (1,9), (1,2), (5,2))$	$(37, 106)$
$Y(8)$	$((1,2), (5,2), (5,9), (8,9), (9,9), (1,9))$	$(37, 105)$
$Y(1)$	$((1,1), (1,2), (1,9), (9,9), (9,5), (7,4))$	$(37, 104)$
$Y(8)$	$((5,2), (1,2), (1,9), (9,9), (8,9), (5,9))$	$(37, 103)$
$Y(7)$	$((1,1), (1,5), (5,5), (8,5), (8,9), (9,9), (1,9), (1,2))$	$(37, 101)$
N		$(38, 98)$
$Y(9)$	$((1,4), (3,6), (4,6), (4,3))$	$(38, 97)$
FN		$(42, 85)$
$Y(3)$	$((1,9), (1,2), (5,2), (6,1), (6,6), (2,6), (2,5), (9,5))$	$(43, 83)$
F		$(81, 0)$

In the protocol of sudoku 147, the ordered pair at the end of the line indicates the number of definitely set digits and the total number of remaining candidates. Among the possible y-chaines, we always select the first among the shortest chaines which has an effect, i.e. leads to elimination of at least one candidate. Thereby, cells are ordered in reading order. First come all cells of the top line of the sudoku, then those of the second, and so on. Chaines of equal length are enumerated according to their first cells, and in each chain, the first cell preceeds the end cell in the order of cells.

As there are often very many y-chaines from which we can choose, we have a huge variety of ways to complete a sudoku like the above one. However, the Y rule is necessary. Without it, sudoku 147 cannot be completed by constraint propagation alone. In cases like this, and the more so in even more complicated cases, it may well be reasonable to include some trial and error.

8 W-Patterns (Swordfish, X-Wing)

For any given digit, the candidates in n given rows occupy at least n columns. Otherwise, some column would necessarily have to contain this digit more than once. If the candidates occupy not more columns as rows, they are said to form a *swordfish*. Then in the solution, the digit will, in these rows, occupy exactly these columns (in whatever order). Therefore in these columns, the digit is not possible outside the given rows.

A swordfish in which, in each of the k mentioned rows, the candidates occupy *all* of the k columns is called X-Wing.

Rule 7 (Swordfisch (W)) `>>sf<<` *If in k rows, the candidates of some given digit occupy just k distinct columns, in these columns the candidates outside the given rows can be eliminated.*

These rules and definitions remain true, of course, if "row" and "column" are interchanged.

Example 8.1 (X-Wing)

If we apply $FNBT_3$ to the sudoku to the right, we get a sudoku with two x-wings with respect to columns. They are illustrated in the sudokus below, together with the effects.

To the left, we see that in columns 3 and 7, candidate 8 is restricted to rows 2 and 8. Therefore, in these rows, candidate 8 can be eliminated outside of columns 3 and 7. To the right, we see that in columns 1, 4, 6, and 9, candidate 1 is restricted to rows 1, 2, 8, and 9. Therefore, in these rows, candidate 1 can be eliminated outside of columns 1, 4, 6, and 9. The sudoku can then be completed by FN.

		2				**6**		
				4				
6			**3**		**2**			**7**
		3		**6**				
	1		**5**	**9**	**3**		**8**	
		9		**1**		**5**		
2			**7**		**8**			**4**
				3				
		7				**9**		

sdk **148**

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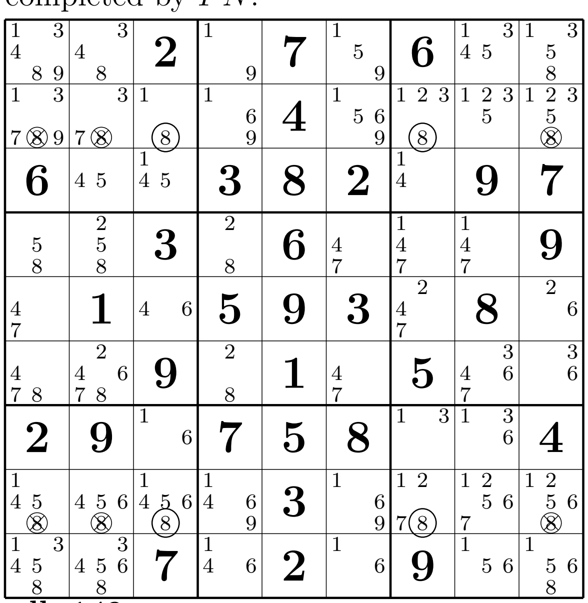
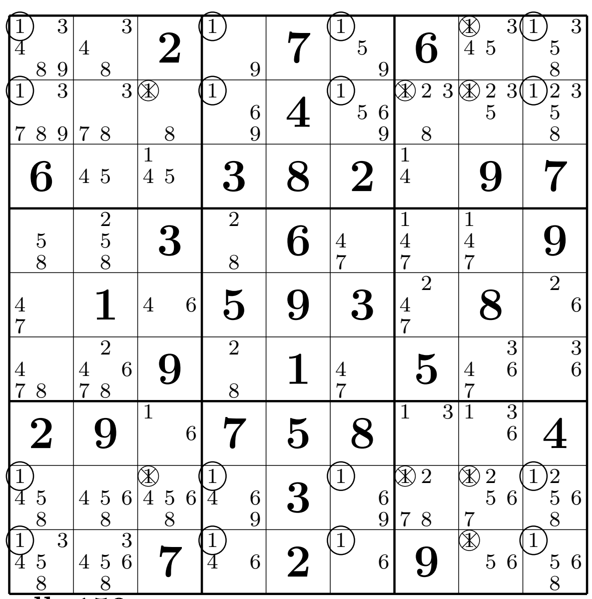


sdk 149 sdk9_tbz_301209_Z49_trsf_e



sdk 150 sdk9_tbz_301209_Z49_trsf_e

8.1 Problems

The following six sudokus can be completed by $FNBT$ and a single application (in one case two applications) of rule W. In fact, in all cases W just means “x-wing”. In some two rows, some candidate is restricted to just two columns.

	1				3	4		
		7	8					1
2				6			9	
8							7	
		9		4		6		
	5							4
	6			5				7
5					1	3		
		3					8	

sdk 151 sdk9_tbz_140410_Z74_trsf

		8	2		7	1		
			8		3			
1								4
3	2						4	6
9	4						5	3
8								1
			7		6			
		7			2	5		

sdk 152 sdk9_tbz_130411_Z74_trsf

Both sudokus could also be completed by $FNBT$ and Y. Furthermore, the first could also be completed by $FNBT$ and X.

		7		5			9	
6				8			3	
			7					6
						6		
3	9			1			5	2
		4						
4					3			
	1			7			8	4
	3			9		1		

sdk 153 sdk9_tbz_180511_Z22_trsf

	8		6		1		7	
9								1
				4				
3				5				4
		4	7		3	2		
1				9				5
				6				
8								2
	6		3		2		8	

sdk 154 sdk9_tbz_111109_Z55_trsf

The first sudoku could also be completed by $FNBT$ and X, the second by$FNBT$ and Y.

3			4		5			7
		2				6		
5				1				9
			6		8			
9				7				4
		3				2		
4			9		7			5

sdk 155 sdk9_NZZaS_210310

	4						5	
2						9		6
	3				8			
	2	7	4		3			
				9				
			8		2	4	3	
			7				9	
7		9	6					5
	1						4	

sdk 156 sdk9_tbz_270711_Z26_trsf

The first sudoku could also be completed with $FNBT$ and both of X and Y instead of W, the second by $FNBT$ and X as well as by by $FNBT$ and Y.

8.2 Hints to the problems

sdk 151 By FNB, the sudoku converts into the state $(45, 92)$ (45 digits definitely set, 92 candidates left). Then in rows 5 and 8, candidate 7 is restricted to columns 2 and 4. As a consequence, candidate 7 can be eliminated from cells $(1, 4)$, $(6, 4)$, $(9, 2)$, and $(9, 4)$. Then completion can be attained by FN.

By $FNBT^2$, we could even reach state $(46, 81)$. Then the x-wing described above would lead to the elimination of candidate 7 from cells $(1, 4)$, $(6, 4)$, and $(9, 2)$, and completion would only require F.

sdk 152 By FNB, we reach state $(53, 70)$. Then in rows 3 and 7, candidate 2 is restricted to columns 3 and 7 and can therefore be eliminated from cells $(2, 7)$, $(5, 7)$, and $(8, 3)$. Then completion can be attained by FN.

sdk 153 By $FNBT^2$ (one hidden pair), we reach state $(29, 179)$. Then in rows 5 and 8, candidate 6 is restricted to columns 3 and 4. Therefore, candidate 6 can be eliminated from cells $(6, 4)$, $(7, 3)$, and $(9, 3)$. Completion is now possible by FNB.

sdk 154 By $FNBT^2$, we reach state $(45, 112)$. We now have even two "x-wings". In rows 1 and 9, candidate 4 is restricted to columns 1 and 7 and can therefore be eliminated from cells $(2, 7)$, $(7, 1)$, $(7, 7)$, and $(8, 7)$. And in the same rows, candidate 5 is restricted to columns 3 and 7, whence it can be eliminated from the 8 cells $(2, 3)$, $(2, 7)$, $(3, 3)$, $(3, 7)$, $(7, 3)$, $(7, 3)$, $(8, 3)$, $(8, 7)$. Completion then only requires F.

sdk 155 By $FNBT_4^2$, we reach state $(46, 102)$. Then in rows 3 and 5, candidate 1 is restricted to columns 1 and 9. It can therefore be eliminated from cells $(7, 9)$, $(8, 1)$, and $(8, 9)$. Then completion can be attained by FN alone.

sdk 156 By $FNBT_3$, we reach state $(54, 67)$. Then in rows 3 and 4, candidate 1 is restricted to columns 5 and 7. It can therefore be eliminated from cells $(1, 7)$, $(5, 7)$, and $(8, 7)$. Completion then only requires FN.

9 Miscellany

9.1 A kind of a meta rule

My collegue Hans Egli, a mathematician from Zürich, drew my attention to a possibility of sudoku completion which is neither based on constraint propagation nor the usual way of trial and error (backtracking). Starting out from the sudoku on the left, by using W and Y as well as $FNBT$, we can get the sudoku on the right. But now, constraint propagation does not lead any further.

	5			4			6	
			9	3	8			
		8				3		
	7	4		9		5	1	
		2				8		
			2	6	3			
	1			7			4	

sdk 157 sdk9_NZZaS_090111_trsf

{2,4,7,9}	{3,8}	{1,7,9}	{1,6,7}	{1,2,5}	{1,2,5,6,7}	{1,2,4,7,9}	{3,8}	{1,2,7,9}
{2,7,8,9}	5	{1,3,7,9}	{1,7}	4	{1,2,7}	{1,2,7,9}	6	{3,8}
{2,4,6,7}	{2,4}	{1,6,7}	9	3	8	{1,2,4,7}	{2,5,7}	{1,2,5,7}
{1,5}	{6,9}	8	{1,4,6,7}	{1,2,5}	{1,2,4,5,6,7}	3	{2,7,9}	{2,4,6,7,9}
3	7	4	8	9	{2,6}	5	1	{2,6}
{1,5}	{6,9}	2	3	{1,5}	{4,6,7}	8	{7,9}	{4,6,7,9}
{4,7,9}	{4,8}	{5,7,9}	2	6	3	{1,7,9}	{5,7,8,9}	{1,5,7,9}
{2,6,8}	1	{3,6}	5	7	9	{2,6}	4	{3,8}
{2,6,7,9}	{2,3}	{5,6,7,9}	{1,4}	8	{1,4}	{6,7,9}	{2,3,5,7,9}	{2,5,7,9}

sdk 158 sdk9_NZZaS_090111e_trsf

In column 5, one of the cells $(1,5)$ and $(4,5)$ finally has to get the digit 2. If we assume this to be cell $(1,5)$, we note the following immediate consequences:

- The candidate set in cell $(4,5)$ reduces to $\{1,5\}$.
- We therefore have an open pair $\{1,5\}$ in row 4, whence candidates 1 and 5 can be eliminated from cell $(4,6)$.

Now in each of the 4 cells $(4,1)$, $(4,5)$, $(6,1)$, and $(6,5)$, we have the same candidate set, which is $\{1,5\}$. In the boxes, rows, and columns which contain theses cells, candidates 1 and 5 do not appear outside of these 4 cells. If in any completion, we interchange 1 and 5 in these 4 cells, the other cells would not be affected. Thus if we had any completion at all, we would have at least two, contradicting the assertion that the sudoku pattern is a sudoku, that is, has exactly one completion. As a consequence, in column 5, we put the 2 into cell $(4,5)$.

Completion now is not simple; but it is possible by constraint propagation. Again, W and Y as well as $FNBT$ are needed. Trial and error usually means propagation to a contradiction, meaning that there is no completion. Here it means propagation to a point where there is more than one completion.

9.2 About diabolical sudoku problems

Sudokus are frequently grouped according to difficulty. It is, however, not advisable to let oneself be overly impressed by terms like "diabolical". In MEPHAM[10], MEPHAM[11], and MEPHAM[12], for instance, there are a total of 36 sudokus described as "diabolical". But they are of widely different degrees of difficulty:

Rules sufficient	Number of sudokus	%
FNB	3	8.3
FNT_2^0	1	2.8
$FNBT_2^0$	2	5.6
$FNBT_2^2 + X$ or $FNBT_2^2 + Y$	4	11.1
$FNB + X$	2	5.6
$FNBT_3^2 + Y$	8	22.2
$FNBT_3^0 + W_2$	1	2.8
$FNBT_3^2 + W_3 + Y$	2	5.6
Trial and error	13	36.1

Of the 36 sudokus, 6 (i. e. 16.7 %) can be completed by $FNBT$ alone.

We find a similar situation in DIE ZEIT/HANDELSBLATT[13]. In the last 36 sudokus, all described as "teuflisch schwer" (devilishly difficult), the methods required are as follows:

Rules sufficient	Number of sudokus	%
FN	4	11.1
FNB or FNT_2^2	3	8.3
FNB	3	8.3
FNT_2^2	4	11.1
$FNBT_2^2 + X$ or $FNBT_2^2 + Y$	5	13.9
$FNBT_2^2 + Y$	12	33.3
$FNB + X_2$	2	5.6
Trial and error	3	8.3

The number of relatively easy problems is considerable: 14 of the 36 sudokus (i. e. 38.9 %) can be completed by $FNBT$ alone, and none of them require both rules B and T. Thereby, open and hidden pairs suffice.

In 10 cases, one single x- or y-chain suffices for completion; only 2 sudokus require for completion more than 4 chains.

10 More on Pair Sequences

>>more<<

10.1 Domino chains

In order to apply rule 6 (Y) in the course of sudoku completion, we have to spot y-chains, and therefore y-sequences. We can, of course, determine by trial and error, whether a given pair sequence is a y-sequence, i.e., whether or not it can be written in the form

$$(\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-2}, c_{n-1}\}, \{c_{n-1}, c_0\}).$$

Trial and error show, for example, that

$$(\{5,7\}, \{4,5\}, \{4,5\}, \{2,5\}, \{1,2\}, \{1,3\}, \{1,3\}, \{1,4\}, \{2,4\}, \{2,7\})$$

is a y-sequence, while

>>notdom<<
$$(\{5,7\}, \{4,5\}, \{4,5\}, \{2,5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \{2,7\}) \quad (2)$$

is not. However, we get a broader view of y-chaines by slightly generalizing the notion:

Definition 14 (Spine, domino sequence) *>>dominochain<<*
Let $\Pi = (\pi_1, \ldots, \pi_n)$ be a pair sequence. We call $s = (c_0, \ldots, c_n)$ a *spine* of Π if s is strictly inhomogeneous and $\pi_i = \{c_{i-1}, c_i\}$ for $i = 1, \ldots, n$. We call Π a *domino sequence* if it has a spine.

If the candidate sets of a cell chain form a pair sequence, then *spine* and *domino* equally apply to the chain. If $c_0 = c_n$, then the domino chain becomes a y-chain. While subsequences of y-sequences are not, in general, y-sequences themselves, they are always domino sequences. The following lemma is a direct consequence of definition 14:

Lemma 10.1 (Subsequence lemma) *>>subsequencelemma<<*

(i) Let $(c_0, \ldots, c_n)$ be a spine, and $\Pi^- = (\pi_i, \ldots, \pi_k)$ $(1 \leq i \leq k \leq n)$ any subsequence, of a pair sequence Π. Then $(c_{i-1}, c_i, \ldots, c_k)$ is a spine of Π^-.

(ii) Any subsequence of a domino sequence is itself a domino sequence.

Lemma 10.2 (Domino lemma) *>>dominolemma<<* *If in a domino chain with spine $(c_0, \ldots, c_n)$ the first cell is* not *assigned candidat c_0, then the last cell is assigned c_n.*

PROOF: By definition 14, if Π is a domino chain with spine $(c_0, \ldots, c_n)$, then it can be written in the form

$$\Pi = (\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}).$$

If the first cell is not assigned candidate c_0, the assignments necessarily are $c_1, c_2, \ldots, c_{n-1}, c_n$. Q.E.D.

The following lemma implies that a pair sequence has at most two distinct spines. If it does have two, both consist of alternating candidates a_1 and b_1, where $\pi_1 = \{a_1, b_1\}$, and the pair sequence is homogeneous. For example, if $\Pi = (\pi_1, \pi_2, \pi_3)$ has two distinct spines, these are (a_1, b_1, a_1, b_1) and (b_1, a_1, b_1, a_1), and therefore $\Pi = (\{a_1, b_1\}, \{a_1, b_1\}, \{a_1, b_1\})$.

Lemma 10.3 (At most two spines) `>>twospine<<` *Let* $(c_0, \ldots, c_n)$ *and* $(d_0, \ldots, d_n)$ *be spines of* $\Pi = (\pi_1, \ldots, \pi_n)$*. Then the following propositions hold:*

(i) If $c_0 = d_0$*, then* $c_i = d_i$ *for* $i = 0, \ldots, n$*.*

(ii) If $c_0 \neq d_0$*, then* $\{c_i, d_i\} = \pi_1$ *for* $i = 0, \ldots, n$*.*

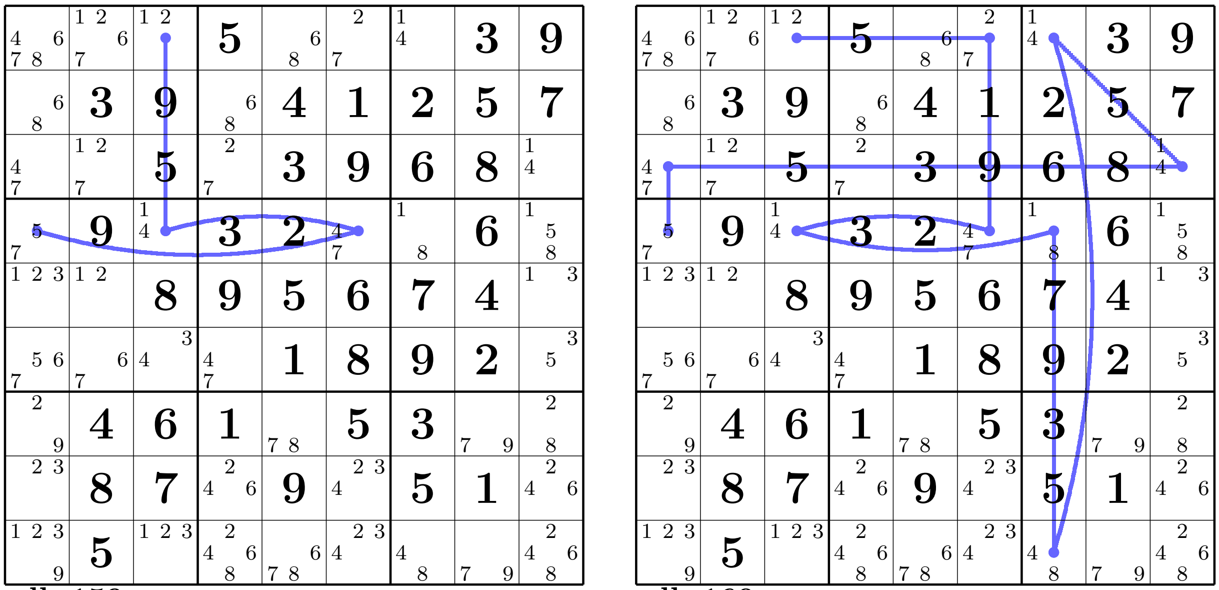
(iii) A pair sequence $\Pi = (\pi_1, \ldots, \pi_n)$ *has two distinct spines if and only if it is homogeneous.*

PROOF: For (i) and (ii), we use induction on n.
(i) If $n > 0$, $c_i = d_i$ for $i = 0, \ldots, n-1$ by the induction hypothesis. From $c_{n-1} = d_{n-1}$ and $\{c_{n-1}, c_n\} = \{d_{n-1}, d_n\} = \pi_n$, it follows that $c_n = d_n$. (Remember that spines are by definition strictly inhomogneous.)
(ii) In case $n = 0$, $\{c_0, d_0\} = \pi_1$ since $c_0, d_0 \in \pi_1$ and $c_0 \neq d_0$. If $n > 0$, $\{c_i, d_i\} = \pi_1$ for $i = 0, \ldots, n-1$ by induction hypothesis. From $\{c_{n-1}, d_{n-1} = \pi_1$, we conclude that $c_{n-1} \neq d_{n-1}$; and therefore from $\{c_{n-1}, c_n\} = \{c_{n-1}, c_n\} = \pi_n$, we get $c_{n-1} = d_n$ and $d_{n-1} = c_n$. Thus $\{c\_n, d_n\} = \{d_{n-1}, c_{n-1} = \pi_1$.
(iii) By (ii), as spines are strictly inhomogeneous, there remain only the possibilities $(a_1, b_1, a_1, \ldots)$ and $(b_1, a_1, b_1, \ldots)$, both spines alternating between a_1 and b_1. Therefore, Π is homogeneous, $\pi_i = \{a_1, b_1\}$ for $i = 1, \ldots, n$. Conversely, if $\Pi = (\pi_1, \ldots, \pi_n)$ is homogeneous, a spine has to alternate between a_1 and b_1. Q.E.D.

While domino chains do not directly lead to the elimination of candidates, they can help to do so indirectly, as the following example will show. In this example, they appear under the name of *forcing chaines*.

Example 10.1 (Forcing chains)

In in PALMSUDOKU[6], *forcing chains* are used to eliminate candidate 7 in cell (4,1):



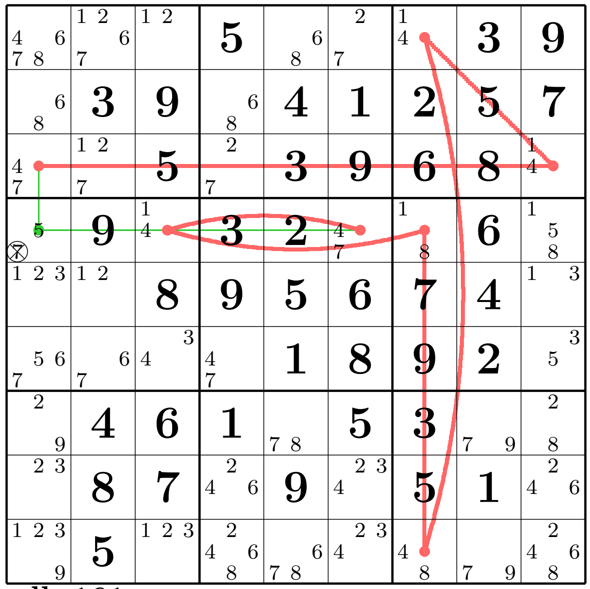
sdk 159 sdk9_palm_1 **sdk 160** sdk9_palm_1

The pictures show two different chains, both beginning at cell (1,3) and ending at cell (4,1). To the path on the left corresponds the pair sequence $(\{2,1\},\{1,4\},\{4,7\},\{7,5\})$, the chain therefore is a domino chain with spine (2, 1, 4, 7, 5). Thus by lemma 10.2, if 2 is *not* assigned to cell (1,3), then 5 is assigned to cell (4,1). The path in the right picture belongs to a domino chain with spine (1, 2, 7, 4, 1, 8, 4, 1, 4, 7, 5), as the corresponding pair sequence is

$$(\{1,2\},\{2,7\},\{7,4\},\{4,1\},\{1,8\},\{8,4\},\{4,1\},\{1,4\},\{4,7\},\{7,5\}).$$

Again by lemma 10.2, if 1 is *not* assigned to cell (1,3), then to cell (4,1) necessarily is assigned 5. As a consequence, to cell (4,1) will in both cases be assigned candidate 5, so candidate 7 can be eliminated.

The same result can be achieved by use of a single y-chain for candidate 7. It is obtained from the long chain on the right by omission of the first two and the last edge, and is a y-chain with respect to candidate 7. It starts in cell (4,6) and ends in cell (3,1). Therefore by the application of this y-chain, candidate 7 can be eliminated from cell (4,1):



sdk 161 sdk9_palm_1

This example does not, of course, prove that the use of forcing chaines can always be replaced with y-chains.

10.2 Choice sequences and signatures

Let $\Pi = (\pi_1, \ldots, \pi_n)$ be any pair sequence, where $\pi_i = \{a_i, b_i\}$, $a_i \neq b_i$ for $i = 1, \ldots, n$.

Definition 15 (Choice sequence) *>>sics<<*
By a *choice sequence* of Π we understand a sequence $(c_1, \ldots, c_n)$ such that $c_i \in \pi_i \quad (i = 1, \ldots, n)$. For a *strictly inhomogeneous choice sequence*, we use the abbreviation *SICS*.

Definition 16 (Signature) *>>sig<<*
We call (d, e) a *signature* of Π if for every SICS $(c_1, \ldots, c_n)$ of Π, $c_1 = d$ or $c_n = e$. By a *signature of a pair chain*, we understand a signature of the corresponding pair sequence.

If $\pi_1 = \{c, d\}$ and $\pi_n = \{e, f\}$, then signature (c, e) excludes SICSs beginning with d and ending on f.
The following lemma is a direct consequence of lemma 10.2:

Lemma 10.4 *>>domsig<<* *A domino sequence with spine* $(c_0, \ldots, c_n)$ *has signature* (c_0, c_n).

Example 10.2

(i) $\Pi = (\{1, 2\}, (\{2, 3\})$. The SICS are $(1, 2)$, $(1, 3)$, and $(2, 3)$. There is exactly one signature, $(1, 3)$, excluding the choice sequence $(2, 2)$, which is not strictly inhomogeneous.

(ii) $\Pi = (\{1,2\},\{1,2\})$. There are exactly 2 SICS: $(1,2)$ and $(2,1)$, and exactly two signatures: $(1,1)$ and $(2,2)$.

(iii) $\Pi = (\{1,2\},\{3,4\})$. All 4 choice sequences are strictly inhomogeneous. As a consequence, Π has no signature.

(iv) $\Pi = (\{1,2\},\{2,3\},\{3,4\})$. The SICS are: $(1,2,3)$, $(1,2,4)$, $(1,3,4)$, and $(2,3,4)$. There is exactly one signature, $(1,4)$, which excludes SICS starting with 2 and ending on 3.

(v) $\Pi = (\{1,2\},\{1,3\},\{1,4\})$. For the SICS (c_1, c_2, c_3), any of the combinations $(c_1, c_3) = (1,1)$, $(1,4)$, $(2,1)$, and $(2,4)$ is possible. Therefore, there is no signature.

(vi) $\Pi = (\{1,2\})$. There are exactly 2 signatures: $(1,2)$ and $(2,1)$. The SICS are (1) and (2). In both cases, $c_1 = c_n = 1$ or $c_1 = c_n = 2$).

The following lemma is a direct consequence of the definitions of y-sequence and signature, and of lemma 10.4:

Lemma 10.5 `>>lemmatransitive<<`

(i) A pair sequence is a y-sequence with respect to candidate c if and only if it has signature (c,c).

(ii) If a pair sequence begins with a subsequence of signature (d,e) and the remaining subsequence has signature (e,f), then the sequence has signature (d,f).

(iii) If a pair sequence has signature (d,e), then the reverse sequence has signature (e,d).

By definition, domino sequences are *contiguous* in the following sense:

Definition 17 (Contiguous pair sequence) `>>contiguous<<`
We say that a pair sequence is *contiguous* if any two consecutive pairs have a candidate in common.

Lemma 10.6 `>>lemmacontinue<<` *We let $\Pi = (\pi_1, \ldots, \pi_n) = (\{a_1, b_1\}, \ldots, \{a_n, b_n\})$ be a pair sequence. Then:*

(i) There exist two SICSs $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_n)$ of Π such that $c_i \neq d_i$ for $i = 1, \ldots n$.

(ii) If (d,e) is a signature of Π, then $d \in \pi_1$ and $e \in \pi_n$

(iii) A homogeneous pair sequence with pairs $\{a,b\}$ has signatures (a,a) and (b,b) if the number of pairs is even; otherwise, it has signatures (a,b) and (b,a).

(iv) *If a pair sequence is not contiguous, then it has no signature.*

(v) *If* $m \leq n$, $\Pi = (\pi_1, \ldots, \pi_n)$ *has not more signatures than* $(\pi_1, \ldots, \pi_m)$.

(vi) *If* Π *is not homogeneous, then it has at most one signature.*

(vii) *A pair sequence has at most two distinct signatures.*

PROOF: (i) By induction on n. $n = 1$: $\pi_1 = \{a_1, b_1\}$. We let $c_1 = a_1$, $d_1 = b_1$. $n > 1$: By induction hypothesis, there are strictly inhomogeneous choice sequences $(c_1, \ldots, c_{n-1})$ and $(d_1, \ldots, d_{n-1})$ for $\Pi = (\pi_1, \ldots, \pi_{n-1})$ such that $c_i \neq d_i$ for $i = 1, \ldots, n-1$. Case (I): $\{c_{n-1}, d_{n-1}\} \cap \pi_n = \emptyset$, where $\pi_n = \{a_n, b_n\}$. Let $c_n = a_n$ and $d_n = b_n$. Case (II): $\{a_{n-1}, b_{n-1}\} = \{c_{n-1}, d_{n-1}\}$ has a common element with $\{a_n, b_n\}$. We may, without loss of generality, assume $c_{n-1} = a_n$. Then $c_{n-1} \neq b_n$ and $d_{n-1} \neq a_n$. Therefore we let $c_n = b_n$ and $d_n = a_n$.

(ii) By (i), there are SICSs $(a_1, \ldots, u)$ and $(b_1, \ldots, v)$ of Π such that $u \neq v$. Now assume that $d \notin \pi_1$. Then $d \neq a_1$, $d \neq b_1$. Because (d, e) supposedly is a signature, this implies $u = e$ and $v = e$, contradicting $u \neq v$. Analogously, the assumption $e \notin \pi_n$ leads to the contradiction $a_1 = b_1 = d$.

(iii) If the number of pairs is even, Π has exactly two SICSs: $(a, b, \ldots, a, b)$ and $(b, a, \ldots, b, a)$. Therefore, the signatures are (a, a) (excluding SICSs beginning and ending on a) and (b, b) (excluding SICSs beginning and ending on b). If the number of pairs is odd, Π has the SICSs $(a, b, \ dots, b, a)$ and $(b, a, \ldots, a, b)$ (and no others). Therefore, the signatures are (a, b) (excluding SICSs beginning with b and ending on a) and (b, a) (excluding SICSs beginning with a and ending on b).

(iv) Assume that π_i and π_{i-1} are disjunct. By (i), $(\pi_1, \ldots, \pi_{i-1})$ has SICSs $(a_1, \ldots, u)$ and $(b_1, \ldots, v)$ such that $\{u, v\} = \pi_{i-1}$. Analogously, $(\pi_n, \ldots, \pi_i)$ has SICSs $(a_n, \ldots, x)$ and $(b_n, \ldots, y)$ such that $\{x, y\} = \pi_i$. Since π_{i-1} and π_i are disjunct, Π has SICS $(a_1, \ldots, u; x, \ldots, a_n), (a_1, \ldots, u; y, \ldots, b_n), (b_1, \ldots, v; x, \ldots, a_n), (b_1, \ldots, v; y, \ldots, b_n)$. Any signature would exclude one of these.

(v) The proof is by induction on $n - m$. If $n - m = 0$, then Π and $(\pi_1, \ldots, \pi_m)$ are identical and therefore have the same signatures. Now assume $n - m > 0$. By induction hypothesis, $\Pi^- = (\pi_1, \ldots, \pi_{n-1})$ has not more signatures than $(\pi_1, \ldots, \pi_m)$. If π_{n-1} and π_n are disjunct, Π has no signature by (iv), and we are done. Otherwise, we have $\pi_n = \{a_{n-1}, c\}$ for some $c \neq a_{n-1}$ or $\pi_n = \{b_{n-1}, d\}$ for some $d \neq b_{n-1}$. (This includes the case where $\pi_n = \pi_{n-1}$.) Each SICS of Π^- has one of the forms

$$(a_1, \ldots, a_{n-1}), (a_1, \ldots, b_{n-1}), (b_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1}).$$

The possible types, concerning the first and the last element, therefore are

$$(a_1, a_{n-1}), (a_1, b_{n-1}), (b_1, a_{n-1}), (b_1, b_{n-1}).$$

If $\pi_n = \{a_{n-1}, c\}$, then we add c to each SICS of Π^- ending on a_{n-1}, and a_{n-1} if it ends on b_{n-1}. Analogously, if $\pi_n = \{b_{n-1}, d\}$, we add b_{n-1} to SICSs of Π^- ending on a_{n-1} and d to those ending on b_{n-1}. Therefore, Π has at least as many types of SICSs as Π^-, and hence not more signatures.

(vi) By induction on n. We let $\Pi^- = (\pi_1, \ldots, \pi_{n-1})$. If Π^- is not homogeneous, then by induction hypothesis and (v), Π has at most one signature. If Π^- is homogeneous, we may assume that $\Pi^- = (\{a,b\}, \ldots, \{a,b\})$ and $\pi_n = \{a,c\}$ for some distinct a, b, c. We now have to distinguish whether $n-1$ is even or odd. If $n-1$ is even, the SICSs of Π are $(a,b, \ldots, a,b,a)$, $(a,b, \ldots, a,b,c)$, $(b,a, \ldots, b,a,c)$. In this case, Π has exactly one signature, (a,c). If $n-1$ is odd, the SICSs of Π are $(a,b, \ldots, b,a,c)$, $(b,a, \ldots, a,b,a)$, $(b,a, \ldots, a,b,c)$. In this case also, Π has exactly one signature, (b,c).
(vii) is a consequence of (iii) and (vi). Q.E.D.

Part (i) of the lemma implies that $(c_i, d_i) = \pi_i$ for $i = 1, \ldots, n$.

Definition 18 (Leading, trailing candidate) *>>leadtrail<<*
We call l a *leading candidate* of Π if $l \in \pi_2 \backslash \pi_1$, and a *trailing candidate* if $t \in \pi_n \backslash \pi_{n-1}$.

Lemma 10.7 *>>leadtrail<<* *Let (d,e) be a signature of Π. Then*

(i) if Π has a trailing candidate, t, then e=t;

(ii) if Π has a leading candidate, l, then d=l;

Proof: (i) Assume that Π has a trailing candidate, t. Then $\pi_{n-1} = \{u,v\}$, $\pi_n = \{v,t\}$ for some u, v, where u, v, t are distinct. By lemma 10.6 (i), there are SICSs $(x, \ldots, u)$ and $(y, \ldots, v)$ of $(\pi_1, \ldots, \pi_{n-1})$ such that $\{x,y\} = \pi_1$ and $\{u,v\} = \pi_{n-1}$. Adding t, we get SICSs $(x, \ldots, u, t)$ and $(y, \ldots, v, t)$ of Π. Therefore, neither of (y,v) and (x,v) can be a signature of Π; whence $e = t$.
(ii) Applying the same argument to the reverse sequence of Π and using lemma 10.5 (iii), we get $e = t$.

Lemma 10.8 *>>redps<<* *Let (c,e) be a signature of $\Pi = (\pi_1, \ldots, \pi_n)$, where $n > 1$, and let $\Pi^- = (\pi_1, \ldots, \pi_{n-1})$. Then if $f \in \pi_n$ and $f \neq e$, (c,f) is a signature of Π^-.*

Proof: By lemma 10.7, $\pi_1 = \{c,d\}$ and $\pi_n = \{e,f\}$ for some d, f such that $d \neq c$, $f \neq e$. By the same lemma, $f \in \pi_{n-1}$, for else f would be the trailing element of Π, which would imply $f = e$. Therefore, $\pi_{n-1} = \{f,g\}$ for some g such that $g \neq f$. (Whether or not $g = e$ is irrelevant.) The assumption that (c,f) is *not* a signature of Π^- would mean that there is a SICS $(d, \ldots, g)$ of Π^-, and therefore a SICS $(d, \ldots, g, f)$ of Π, contradicting the assumption that (c,e) is a signature of Π. Q.E.D.

Lemma 10.9 (Signature and spine) *>>sigspine<<*

(i) Let $\Pi = (\pi_1, \ldots, \pi_n)$ be a pair sequence. Then (c,e) is a signature of Π if and only if there is a spine $(c_0, \ldots, c_n)$ of Π such that $c_0 = c$ and $c_n = e$.

(ii) If $\Pi = (\pi_1, \ldots, \pi_n)$ has a subsequence $\Pi^- = (\pi_i, \pi_{i+1}, \ldots, \pi_k)$ without signature, then Π has no signature.

Proof: (i) If $(c_0, \ldots, c_n)$ is a spine of Π, where $c_0 = c$ and $c_n = e$, then (c, e) is a signature by lemma 10.4.

Conversely, let (c, e) be a signature. The proof is by induction on n. (A) If $n = 1$, signatures and spines are identical, namely (c, e) and (e, c). Therefore, (c, e) is a spine. (B) If $n > 1$, we let $\Pi^- = (\pi_1, \ldots, \pi_{n-1})$. Then by lemma 10.8, Π^- has a signature (c, f) where $f \in \pi_n$ and $f \neq e$. By induction hypothesis, Π^- has a spine $(c_0, \ldots, c_{n-1})$ such that $c_0 = c$, $c_{n-1} = f$. Then $\pi_n = \{c_{n-1}, e\} = \{f, e\}$, whence $(c_0, \ldots, c_{n-1}, e)$ is a spine of Π.

(ii) Assume that Π has a signature. By (i), it then has a spine $(c_0, \ldots, c_n)$. Therefore, $(c_{i-1}, c_i, \ldots, c_k)$ is a spine, and (c_{i-1}, c_k) is a signature, of Π^-, contradicting the assumption. Q.E.D.

Definition 19 (Rotten triple) `>>rotten<<`
By a *rotten triple* we mean a contiguous, strictly inhomogeneous sequence of three pairs which contain a common candidate.

Lemma 10.10 `>>rotlem<<`

(i) A pair sequence containing a rotten triple has no signature.

(ii) A contiguous, strictly inhomogeneous pair sequence not containing a rotten triple has a signature.

Proof: (i) Let $\Pi = \{a, u\}, \{a, v\}, \{a, w\})$ be a rotten triple. Then $u \neq v$ and $v \neq w$ (while $u = w$ is admitted). Then Π has SICSs (a, v, a), (a, v, w), (u, a, w), and (u, v, a), excluding a signature. If Π is any pair sequence containing a rotten triple, it has no signature by lemma 10.9 (ii).

(ii) The proof is by induction on the number of pairs of $\Pi = (\pi_1, \ldots, \pi_n)$.

(A) $n < 3$. If $\Pi = (\{a_1, b_1\})$, the signatures are (a_1, b_1) and (b_1, a_1). If $\Pi = (\{a_1, b_1\}, \{a_2, b_2\})$, Π has one or two signatures, depending on whether $\pi_1 \neq \pi_2$ or not.

(B) $n \geq 3$. We let $\Pi = (\pi_1, \ldots, \pi_{n-2}, \pi_{n-1}, \pi_n)$ and $\Pi^- = (\pi_1, \ldots, \pi_{n-2}, \pi_{n-1})$. By induction hypothesis, Π^- has a signature (c, e) and therefore, by lemma 10.9 (i), a spine $(c_0, \ldots, c_n)$ $(c_0, \ldots, c_{n-2}, c_{n-1})$ such that $c_0 = c$, $c_{n-1} = e$. Hence $\pi_{n-2} = \{c_{n-3}, c_{n-2}\}$, $\pi_{n-1} = \{c_{n-2}, c_{n-1}\}$. As Π is strictly inhomogeneous and does not contain a rotten triple, $c_{n-3} \neq c_{n-1}$ and $c_{n-2} \notin \pi_n$. Since Π is contiguous, / $c_{n-1} \in \pi_n$, whence $\pi_n = \{c_{n-1}, a\}$ for some $a \neq c_{n-1}$. So $(c_0, \ldots, c_{n-2}, c_{n-1}, a)$ is a spine, and (c_0, a) a signature, of Π. Q.E.D.

Note that (ii) does not hold if we drop the restriction to strictly inhomogeneous pair sequences: $(\{1, 3\}, \{1, 2\}, \{1, 2\}, \{2, 4\})$ does not contain a rotten triple. It has, however, no signature, as is shown by the SICSs $(1, 2, 1, 2)$, $(1, 2, 1, 4)$, $(3, 1, 2, 4)$, $(3, 2, 1, 2)$.

Lemma 10.11 (Splitting) `>>split<<`
If (d, e) is a signature of $\Pi = (\pi_1, \ldots, \pi_n)$, and if $\Pi^- = (\pi_i, \ldots, \pi_i)$, $\Pi^+ = (\pi_{i+1}, \ldots, \pi_n)$ where $1 \leq i < n$, then for some f, (d, f) and (f, e) are signatures of Π^- and Π^+, respectively.

Proof: By lemma 10.9, Π has a spine $(c_0, \ldots, c_n)$ such that $c_0 = d$ and $c_n = e$. Then $(c_0, \ldots, c_i)$ and $(c_i, \ldots, c_n)$ are spines of Π^- and Π^+, respectively. Therefore by lemma 10.4, (c_0, c_i) and (c_i, c_n) are signatures of Π^- and Π^+, respectively. Thus c_i is the desired f. Q.E.D.

10.3 A decision method for domino sequences

In view of lemma 10.5 (ii) and lemma 10.11 we can decide whether any given pair sequence Π is a domino sequence (has a signature) in the following manner:

1. Partition Π into
   - homogeneous subsequences of maximum length and
   - strictly inhomogeneous subsequences.

   (This partitioning is unique.) Note that pair sequences of length $n = 1$ are homogeneous.
2. If any of the strictly inhomogeneous subsequences contains a rotten triple, then Π is not a domino sequence (has no signature).
3. Otherwise, from each of the homogeneous subsequences we can choose between the two signatures. Now Π is a domino sequence if and only if these choices can be made in such a way that for any two consecutive signatures, the first ends with the candidate that the second starts with.

For example, the pair sequence (1) of section 7:

$$(\{5,7\}, \{7,1\}, \{1,6\}, \{6,1\}, \{1,6\}, \{6,1\}, \{1,9\}, \{9,3\}, \{3,9\}, \{9,7\},$$
$$\{7,9\}, \{9,8\}, \{8,2\}, \{2,3\}, \{3,8\}, \{8,3\}, \{3,1\}, \{1,4\}, \{4,8\}, \{8,5\})$$

can be partitioned as follows: $(\{5,7\}, \{7,1\})$ / $(\{1,6\}, \{6,1\}, \{1,6\}, \{6,1\})$ / $(\{1,9\})$ / $(\{9,3\}, \{9,3\})$ / $(\{9,7\}, \{7,9\})$ / $(\{9,8\}, \{8,2\}, \{2,3\})$ / $(\{3,8\}, \{8,3\})$ / $(\{3,1\}, \{1,4\}, \{4,8\}, \{8,5\})$.

This gives us the following sequence of signatures: $(5,1)$, $\frac{(1,1)}{(6,6)}$, $\frac{(1,9)}{(9,1)}$, $\frac{(3,3)}{(9,9)}$, $\frac{(7,7)}{(9,9)}$, $(9,3)$, $\frac{(3,3)}{(8,8)}$, $(3,5)$. By choosing where we have two possibilities, we get the signatures $(5,1)$, $(1,1)$, $(1,9)$, $(9,9)$, $(9,9)$, $(9,3)$, $(3,3)$, $(3,5)$ for the subsequences, and therefore (by lemma 10.5 (ii)) the signature $(5,5)$ for the pair sequence (1). Hence we have a y-sequence for candidate 5.

If, for instance, we would omit one of the pairs $\{1,6\}$, the corresponding signatures $\frac{(1,1)}{(6,6)}$ would turn into $\frac{(1,6)}{(6,1)}$; therefore, Π would cease to be a domino chain.

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